Weighted homogeneous singularities and rational homology disk smoothings

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Abstract We classify the resolution graphs of weighted homogeneous surface singularities which admit rational homology disk smoothings. The nonexistence of rational homology disk smoothings is shown by symplectic geometric methods, while the existence is verified via smoothings of negative weights. In particular, it is shown that a (negative definite) starshaped plumbing tree gives rise to a weighted homogeneous singularity admitting a rational homology disk smoothing if and only if the Milnor fillable contact structure of the link admits a weak symplectic rational homology disk filling.

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1 Introduction

The rational blow-down procedure (introduced by Fintushel and Stern [5] and extended by Park [14]) turned out to be one of the most effective operations for constructing exotic smooth 4-manifolds. In this procedure the tubular neighbourhood of a collection of 2-spheres — with intersection pattern given by a linear plumbing tree, with framings given by the continued fraction coefficients of $-\frac{p^2}{pq-1}$ for some relatively prime $p > q > 0$ — is replaced by a rational homology disk, i.e., with a 4-manifold with boundary which has rational homology isomorphic to $H_*(D^4;\mathbb{Q})$. (Let $\mathcal{G}$ denote the set of all linear plumbing chains considered above.) It was a natural question to seek a generalization of this method for other plumbing trees. Seiberg-Witten theoretic considerations suggested that one should focus on negative definite plumbing trees (which therefore give rise to surface singularities) and require that the rational homology disk is a smoothing of the singularity. In [17] the restrictions on the combinatorics of the plumbing tree implied by the existence of such a smoothing were explored. The graphs satisfying the combinatorial constraints have been identified, but the question of which graphs actually give rise to singularities admitting rational homology disk smoothing has been left open.

The link Y_Γ of a singularity S_Γ with resolution graph Γ is determined by the

plumbing graph, and according to [3] the 3-manifold Y_Γ admits a (up to contactomorphism) unique contact structure, its *Milnor fillable contact structure* ξ_Γ given by the 2-plane field of complex tangencies on Y_Γ as a link of S_Γ. Any smoothing of the singularity S_Γ provides a Stein filling of the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ). In [17] the more general question of exploring the existence of weak symplectic rational homology disk fillings of the Milnor fillable contact structures on minimal negative definite plumbing trees has been treated, and the same conclusion has been drawn for the combinatorics of these trees as for surface singularities with rational homology disk smoothings. The complete answer for the geometric question remained open.

In this paper we provide the complete classification of those plumbing trees which are minimal, negative definite, starshaped, the central vertex v has framing at most one less in absolute value than its valency and (a) there is a surface singularity with this given resolution graph which admits rational homology disk smoothing, or (b) the Milnor fillable contact structure (Y_Γ, ξ_Γ) corresponding to the plumbing tree admits a weak symplectic rational homology disk filling. To state the precise result, we need a few definitions.

Definition 1.1 A singularity S_Γ is called *Seifert* if the link of the singularity is a Seifert fibered 3-manifold over the sphere. S_Γ is *small Seifert* if the link is a small Seifert fibered 3-manifold, i.e., it admits a Seifert fibration over S^2 with exactly three singular fibers.

A singularity is therefore Seifert if and only if it admits a resolution graph which is a starshaped tree and the vertices correspond to rational curves; in addition, S_Γ is small Seifert if the central vertex (the unique vertex of valency > 2) in a minimal good resolution is of valency 3. By [13, Theorem 2.6.1] members of a well-studied class of singularities, the *weighted homogeneous* singularities with rational homology sphere links are all Seifert singularities. (For a definition of weighted homogeneous singularities see, for example, [13, p. 206].)

Definition 1.2 Define $\mathcal{QHD}_3$ as the set of all graphs given by Figures 1(a) through (g) and Figures 2(a) through (f). In Figure 1(a) $p, q, r \geq 0$, in (b) $p \geq 1$, $q, r \geq 0$, in (c) $q, r \geq 0$, in (d) $r \geq 1$, $q \geq 0$, in (e) $p \geq 1$, $q \geq 0$, in (f) $q \geq 0$ while in (g) $p, r \geq 1$ and $q \geq 0$. In Figure 2 $n \geq 2$ for (a), (b) and (c) and $n \geq 1$ for (d), (e) and (f).

Remark 1.3 The graphs given in Figure 1(a) form the set $\mathcal{W}$ of [17]; Figures 1(b) and (c) form $\mathcal{N}$ while the collection of (d), (e), (f) and (g) were called $\mathcal{M}$ in [17]. The graphs of Figure 2(a) are in the class $\mathcal{A}$ of [17], the ones of the

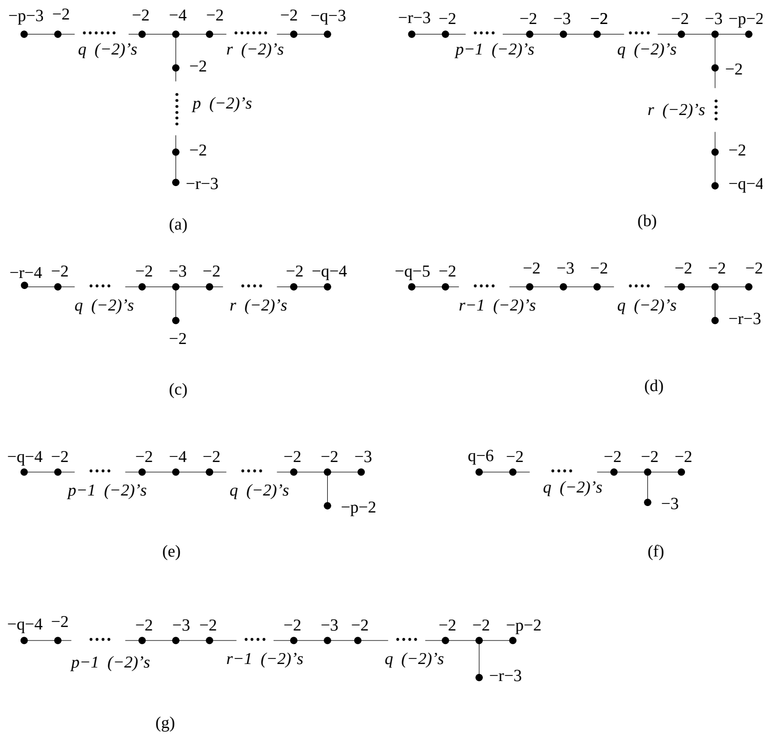

Figure 1: The graphs defining the class $\mathcal{QHD}_3$ of plumbing graphs. In (a) $p,q,r \geq 0$, in (b) $p \geq 1,\ q,r \geq 0$, in (c) $q,r \geq 0$, in (d) $r \geq 1,\ q \geq 0$, in (e) $p \geq 1,\ q \geq 0$, in (f) $q \geq 0$ while in (g) $p,r \geq 1$ and $q \geq 0$.

form (b) and (c) are in $\mathcal{B}$ and (d), (e) and (f) are in $\mathcal{C}$. (For the definition of these classes of graphs see Subsection 2.2.)

According to [6] singularities corresponding to the resolution trees in $\mathcal{QHD}_3$ are all *taut*, that is, the resolution graph uniquely determines the analytic structure of the singularity. Since for any starshaped negative definite plumbing tree of spheres there is a weighted homogeneous singularity with that resolution graph [15, Theorem 2.1], the unique singularity above is necessarily weighted homogeneous. With this terminology in place, the first main result of the paper is

Theorem 1.4 *Suppose that S_Γ is a small Seifert singularity with link Y_Γ. Assume that Γ is a minimal good resolution graph of S_Γ, which is therefore a negative definite tree with three branches. Then the following three statements*

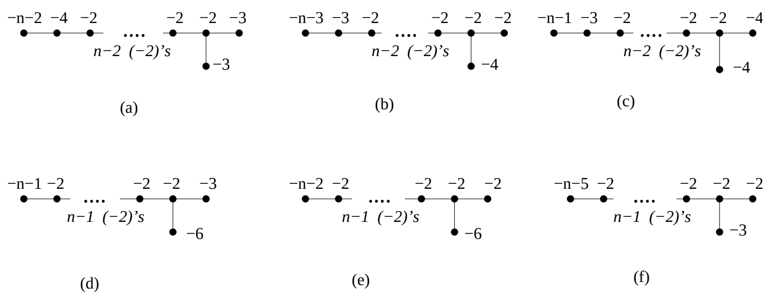

Figure 2: The graphs defining the class $\mathcal{QHD}_3$ of plumbing graphs. We require $n \geq 2$ for (a), (b) and (c) and $n \geq 1$ for (d), (e) and (f).

are equivalent:

(1) *The singularity S_Γ admits a rational homology disk smoothing.*

(2) *The Milnor fillable contact structure on Y_Γ admits a weak symplectic rational homology disk filling.*

(3) *The graph Γ is in $\mathcal{QHD}_3$.*

For starshaped diagrams with more than three branches the analytic type of the singularity typically is not determined by the graph itself, hence the formulation of our result needs a little more care.

Definition 1.5 Define $\mathcal{QHD}_4$ as the union of all graphs given by Figure 3(a), (b) and (c) for $n \geq 2$ in each case.

With this notation in place, we are ready to state the second main result of the paper:

Theorem 1.6 *Suppose that Γ is a minimal, starshaped plumbing tree with at least four branches, and the framing of the central vertex is less than -2. Then the following statements are equivalent.*

(1) *There is a Seifert singularity S_Γ with resolution graph Γ which admits a rational homology disk smoothing;*

(2) *The Milnor fillable contact structure on Y_Γ admits a weak symplectic rational homology disk filling; and*

(3) *The graph Γ is in $\mathcal{QHD}_4$.*

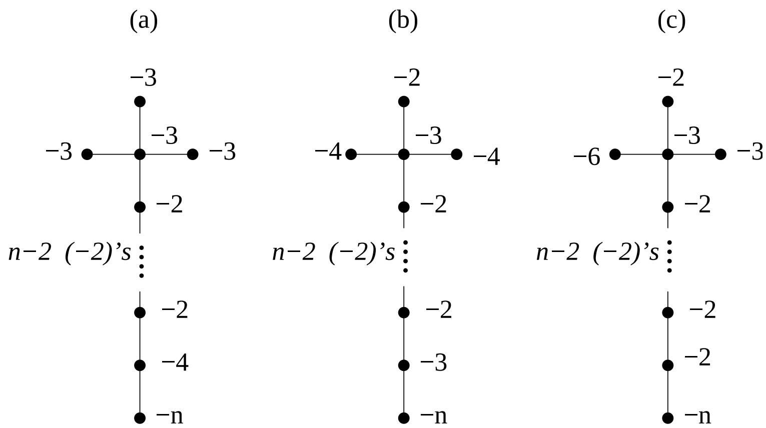

Figure 3: The graphs (with $n \geq 2$) defining the class $\mathcal{QHD}_4$ of plumbing graphs.

Remarks 1.7 **(a)** The assumption on the framing of the central vertex is not essential in the singularity theoretic part of the theorem: a surface singularity with rational homology disk smoothing must be rational, hence the assumption on the central framing should be satisfied. Consequently, the equivalence of (1) and (3) holds without the assumption on the central framing. In the symplectic topological result (regarding the rational homology disk fillings of the Milnor fillable contact structure), however, our methods do not work unless the additional hypothesis on the central framing is assumed. It is reasonable to expect, though, that the Milnor fillable contact structures on 3-manifolds defined by negative definite four-legged plumbing trees with central framing (-2) do not admit rational homology disk weak fillings.

(b) According to [6] the analytic type of the singularity S_Γ with $\Gamma \in \mathcal{QHD}_4$ is determined by the analytic type of the central curve in the resolution. It is still an open question whether for a fixed $\Gamma \in \mathcal{QHD}_4$ there is a unique singularity with the given resolution graph admitting a rational homology disk smoothing, or there are more than one analytically distinct such singularities.

(c) Combining the result of [6] with [15, Theorem 2.1] (and the text following the theorem) we get that any singularity with resolution graph $\Gamma \in \mathcal{QHD}_4$ is weighted homogeneous.

A possible interpretation of Theorems 1.4 and 1.6 is that for weighted homogeneous singularities smoothing theory and symplectic topology behave in a parallel manner, at least as far as existence of rational homology disk fill-

ing/smoothing goes. This interpretation fits in the line of current results; notice the similarity with the result of Némethi and Popescu-Pampu [11], where a natural bijection between smoothings and minimal symplectic fillings of cyclic quotient singularities has been established.

The idea of the proof of the main results can be summarized as follows. Recall from [17] that if a starshaped graph defines a singularity with rational homology disk smoothing, or gives rise to a Milnor fillable contact structure with a weak symplectic rational homology disk filling, then the valency of the central vertex is 3 or 4. (For a precise formulation of this result, see Theorem 2.9.) If this valency is 3, then there are three triply infinite families (called $\mathcal{W}, \mathcal{M}$ and $\mathcal{N}$ in [17], cf. Remark 1.3) where each member defines a singularity with the required smoothing. In addition, there are three further families ($\mathcal{A}, \mathcal{B}, \mathcal{C}$ in [17], cf. also Subsection 2.2) which contain all further such 3-legged graphs. We will systematically examine all 3-legged elements of these further families, show that for most of them the associated Milnor fillable contact structure does not admit a weak symplectic rational homology disk filling, and for those we cannot exclude the existence of such a filling, we construct the rational homology disk smoothing of the singularity. The nonexistence proofs rely on symplectic geometric results (notably on McDuff's result regarding symplectic manifolds containing symplectic spheres of self-intersection $(+1)$) and tedious combinatorial arguments. In principle these arguments could be extended to cover other symplectic fillings, but the combinatorics (which is already quite delicate for the case of rational homology disk fillings) can become too complex to handle. In proving the existence of the smoothing we will apply a result of Pinkham, formulated in Theorem 2.12, cf. also [17, Section 8.1]. In the 4-legged case we only have to examine the families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, and the adaptation of the same strategy above, in fact, provides the result.

The paper is organized as follows. In Section 2 the symplectic geometric preliminaries used in the proofs of the main results are listed, together with a quick outline of the ideas employed in the later arguments. Section 3 deals with small Seifert singularities, i.e. with those singularities which have starshaped minimal good resolution graphs with three branches. In Section 4 we address the general case of Seifert singularities. Finally in Section 5 (for the sake of completeness) we recall the existence of the smoothings for graphs in classes $\mathcal{W}, \mathcal{M}, \mathcal{N}$, which were already discussed in [17].

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2 Preliminaries

2.1 Symplectic geometric preliminaries

Our results rely on the following fundamental theorem due to McDuff.

Theorem 2.1 *(McDuff, [8, Theorem 1.4]) Let (M, ω) be a closed symplectic 4-manifold. If M contains a symplectically embedded 2-sphere L of self-intersection number 1, then M is a rational symplectic 4-manifold. In particular, M becomes a the complex projective plane after blowing down a finite collection of symplectic (-1)-curves away from L.* □

Remark 2.2 Since a symplectic $(+1)$-sphere in a symplectic 4-manifold admits a concave neighborhood, the above statement is equivalent to the fact that the unique tight contact structure ξ_{st} on the 3-sphere S^3 admits a unique minimal symplectic filling, which is diffeomorphic to the 4-disk [4]. In the present context the form given by Theorem 2.1 is more convenient for us, since it allows to consider curves intersecting L in M.

The following two lemmas are based on the above theorem of McDuff and are proved in [1]:

Lemma 2.3 *([1, Lemma 2.13]) Let (M, ω) be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1 and a collection of symplectically immersed 2-spheres $C_1, \dots, C_k$. Suppose that J is a tame almost complex structure for which L, $C_1, \dots, C_k$ are pseudoholomorphic. Then there exists at least one J-holomorphic (-1)-curve in $M - L$ unless $L \cdot C_i > 0$ and $C_i \cdot C_i = (L \cdot C_i)^2$ for all i.* □

Lemma 2.4 *([1, Lemma 2.5]) Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. If C is an irreducible singular or higher genus pseudoholomorphic curve in M, then $C \cdot L \geq 3$. In particular there are no irreducible singular or higher genus pseudoholomorphic curves in $M - L$.* □

This lemma has the following simple corollary.

Corollary 2.5 *Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. Then there is no cycle of pseudoholomorphic spheres in the complement L.*

Proof If such a cycle existed, by gluing adjacent components around the nodes we would be able to construct an embedded pseudoholomorphic curve of genus 1 which would contradict Lemma 2.4. □

Another fact which we will frequently use is that for any almost complex structure J on a 4-manifold X any intersection point of two J-holomorphic curves C_1 and C_2 contributes positively to the algebraic intersection number $C_1 \cdot C_2$. This result of McDuff is quite simple for transverse intersections, but requires a delicate argument (due to McDuff) for the general case (see [9], [10]). In our subsequent arguments we will use only the transverse case.

The next lemma easily follows from McDuff's Theorem 2.1.

Lemma 2.6 *Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. Then there is no symplectically embedded sphere of nonnegative self intersection number in the complement of L.*

Proof Since M is rational, it follows that $b_2^+(M) = 1$, immediately implying the lemma. (Notice that a symplectic sphere of any self-intersection — including 0 — is homologically essential.) □

Lemma 2.7 *Suppose that $C \subset \mathbb{CP}^2$ is a J-holomorphic curve for some tame almost complex structure J, in the homology class $[C] = d[\mathbb{CP}^1]$, and C has at least two singular points. Then $d \geq 4$.*

Proof The J-holomorphic line passing through two singular points intersects C with multiplicity at least 4, providing the result. □

We record here the following fact which we will apply repeatedly in the sequel: By the adjunction formula, a pseudoholomorphic rational curve representing the class $3[\mathbb{CP}^1]$ in $\mathbb{CP}^2$ must be either immersed with exactly one node (that is a point where two branches of the curve intersect transversely) or it must have exactly one nonimmersed point which is necessarily a $(2,3)$-cusp singularity. (Here a pseudoholomorphic curve in a 4-manifold is said to have a $(2,3)$-cusp singularity if there is a parametrization around the singular point in which the

curve has the form $(z^2, z^3) + O(4)$, see [9].) In conclusion, the link of such a curve around its singular point is either connected (and is the trefoil knot) or has two components (and is the Hopf link).

2.2 The families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$

The three inductively defined families $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of graphs found in [17] will play a cental role in our subsequent arguments. For the sake of completeness, we recall the definition of these families below.

Let us define $\mathcal{A}$ as the family of graphs we get in the following way: start with the graph of Figure 4(a), blow up its (-1)-vertex or any edge emanating from the (-1)-vertex and repeat this procedure of blowing up (either the new (-1)-vertex or an edge emanating from it) finitely many times, and finally modify the single (-1)-decoration to (-4). Depending on the number and configuration of the chosen blow-ups, this procedure defines an infinite family of graphs. Define $\mathcal{B}$ similarly, this time starting with Figure 4(b) and substituting (-1) in the last step with (-3), and finally define $\mathcal{C}$ in the same vein by starting with Figure 4(c) and putting (-2) in the place of (-1) in the final step.

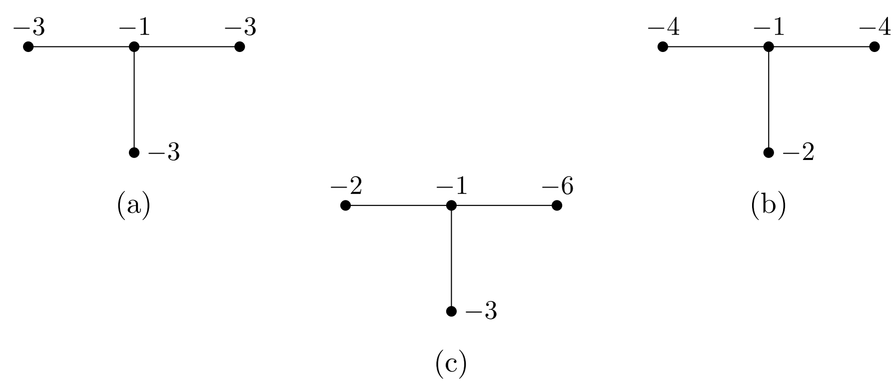

Figure 4: Nonminimal plumbing trees giving rise to the families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

Remark 2.8 Figure 5 gives a pictorial description of what we mean by blowing up a (-1)-vertex (Figure 5(a)) and an edge emanating from a (-1)-vertex (Figure 5(b)). Notice that in the plumbing 4-manifold both operations correspond to blowing up the (-1)-sphere defined by the vertex, either in a generic

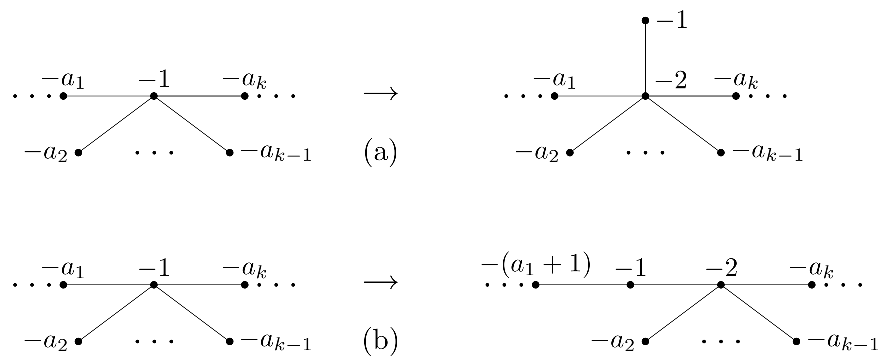

Figure 5: The blow-up of (a) a (-1)-vertex and (b) an edge emanating from a (-1)-vertex.

point or in an intersection with another sphere of the plumbing configuration. A graph in $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ is a starshaped three-legged graph if and only if in the defining procedure we always blow up edges (and never vertices), and is four-legged if and only if we start by blowing up the central vertex, and then never blow up a vertex after we blew up an edge. That is, we blow up vertices n times, and then we only blow up edges. Notice that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ contains no starshaped graph of more than four legs.

The starting point of the proofs of Theorems 1.4 and 1.6 rests on the main result of [17] which can be summarized as follows. With the definition of $\mathcal{W}, \mathcal{M}, \mathcal{N}$ given in Remark 1.3 and of $\mathcal{G}$ given in the first paragraph of Section 1, we have

Theorem 2.9 *([17]) Suppose that Γ is a minimal, negative definite plumbing tree. If it gives rise to a surface singularity S_Γ admitting a rational homology disk smoothing, or if the Milnor fillable contact structure on the corresponding plumbing 3-manifold Y_Γ admits a rational homology disk filling then Γ is in $\mathcal{G} \cup \mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.* $\square$

Remark 2.10 The method of the proof of the main results of the present paper make use of the fact that the singularities under examination are Seifert, that is, the resolution graphs are starshaped. The existence question of rational homology disk smoothings/fillings for singularities with non-starshaped resolution graphs is still open; we hope to return to this question in a future project.

2.3 Outline of the proofs

The heart of the proofs of Theorem 1.4 and Theorem 1.6 is the implication (2) $\Rightarrow$ (3) in each case. The strategy in both cases is as follows.

Suppose that Γ is a graph of the type considered in Theorem 1.4 or Theorem 1.6. Let Y_Γ denote the associated plumbed 3-manifold and ξ_Γ the unique Milnor fillable contact structure on Y_Γ. According to Theorem 2.9, if (Y_Γ, ξ_Γ) admits a symplectic rational homology disk filling then Γ must be in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Since the singularities corresponding to graphs in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$ admit rational homology disk smoothings (cf. [17, Section 8] or Section 5), the corresponding links admit symplectic rational homology disk fillings. Hence we only need to consider graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ satisfying the hypotheses of Theorem 1.4 or Theorem 1.6.

Let Γ be a graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ satisfying the hypotheses of Theorem 1.4 or Theorem 1.6. The first step is to find an appropriate strong concave filling of (Y_Γ, ξ_Γ). To find such a concave filling, we apply a standard topological construction, which in fact applies for any negative definite starshaped plumbing graph and which we recall presently. Suppose that Γ is a negative definite starshaped plumbing graph with s legs $\ell_1, \ldots, \ell_s$ and with central framing b. Suppose that the framing coefficients along the leg ℓ_i are given by the continued fraction coefficients of $-\frac{n_i}{m_i} < -1$. Consider then the "dual" graph Γ' which is starshaped with s legs $\ell'_1, \ldots, \ell'_s$, central framing $-b-s$, and the framings along the leg ℓ'_i are given by the continued fraction coefficients of $-\frac{n_i}{n_i - m_i}$. Let W_Γ and $W_{\Gamma'}$ denote the corresponding plumbing 4-manifolds. In the following lemma we formulate a well-known simple fact, cf. also [7, 17].

Lemma 2.11 *Suppose that Γ is a negative definite starshaped plumbing tree, and Γ' is its dual tree constructed above. The boundary of W_Γ is orientation preserving diffeomorphic to the link Y_Γ while $\partial W_{\Gamma'} = -Y_\Gamma$. In addition, $W_\Gamma \cup W_{\Gamma'}$ is a 4-manifold diffeomorphic to $\mathbb{CP}^2 \# m \overline{\mathbb{CP}^2}$ for some positive integer m.*

Proof (sketch) Consider the Hirzebruch surface with zero-section of self-intersection b (and hence with infinity-section of self-intersection $-b$). Fix s distinct fibers of the $\mathbb{CP}^1$-fibration and blow up the intersection points of these fibers with the infinity-section. After the appropriate repeated blow-ups we can identify in the resulting rational surface a configuration of curves intersecting each other according to Γ, and it is easy to see that the complementary curves will intersect each other according to Γ'. Since the curves intersecting according to the graph Γ admit a strong convex neighbourhood, with the

Milnor fillable contact structure as induced structure on the boundary, the complement (diffeomorphic to $W_{\Gamma'}$) provides a strong concave filling of (Y_Γ, ξ_Γ). Since the complement is also a tubular neighbourhood of a configuration K of curves (intersecting each other according to Γ'), we will refer to K as the *compactifying divisor.* □

Suppose now that X is a weak symplectic rational homology disk filling of (Y_Γ, ξ_Γ). Since Y_Γ is a rational homology 3-sphere, we can perturb the symplectic structure on X in a neighbourhood of the boundary so that it becomes a strong symplectic filling of (Y_Γ, ξ_Γ). Glue X and $W_{\Gamma'}$ along Y_Γ to obtain a closed symplectic 4-manifold Z. Notice that this is the point where symplectic methods do apply, while holomorphic techniques do not necessarily work anymore: by gluing the filling (even if it admits complex analytic structures) to the compactifying divisor we cannot necessarily glue the complex structures together.

Let k denote the number of irreducible components of the compactifying divisor K. Then since $W_{\Gamma'}$ is a regular neighbourhood of K, we have that $b_2(W_{\Gamma'}) = k$. Since X is a rational homology disk, it follows that $b_2(Z) = k$.

In all cases that we consider, it turns out that K (after, possibly, a sequence of blow-downs) contains a component which is a sphere that is embedded in $W_{\Gamma'} \subset Z$ with self intersection number $(+1)$. Let L denote one such component. By McDuff's Theorem 2.1, we conclude that Z is a rational symplectic 4-manifold and hence diffeomorphic to $\mathbb{CP}^2 \# (k-1)\overline{\mathbb{CP}^2}$. By McDuff's Theorem, for a generic tame almost complex structure J, in the complement of L we can find $k-1$ disjoint embedded symplectic 2-spheres with self-intersection number -1 (we will refer to these as *symplectic* (-1)*-curves*), and after blowing these down we obtain $\mathbb{CP}^2$. However, we would like to understand how the other components of K descend under the blowing down map. We thus proceed as follows.

We choose a tame almost complex structure J on Z with respect to which all the curves in K are pseudoholomorphic. (In fact, we can assume that J is *integrable* on $W_{\Gamma'}$.) We assume that J is generic among those almost complex structures for which K is J-holomorphic. Appealing to Lemma 2.3 we can find a pseudoholomorphic (-1)-curve E in Z disjoint from L. By perturbing the almost complex structure J if necessary, we can assume that E intersects each component of K transversely and does not pass through any point where two or more components of K pass. We then blow down E. By [12, Lemma 4.1] we can find a tame almost complex structure J' on the blown

down manifold Z' with respect to which the images of all the components of K are pseudoholomorphic. We will again be in the situation where we have a closed symplectic 4-manifold containing a symplectically embedded 2-sphere of self-intersection number 1 and a collection of symplectically immersed 2-spheres (the images of the components of $K - L$). We can thus again appeal to Lemma 2.3 and find a pseudoholomorphic (-1)-curve E' in Z'. Note that E' may be a component of K', the image of the configuration K under the blowing down map. By suitably perturbing the almost complex structure, we can arrange that E' intersects each component of $K' - E'$ transversely and it does not pass through any point where two or more components of $K' - E'$ pass. We then blow down E'. Proceeding in this way, repeatedly blowing down (-1)-curves whose existence is given by Lemma 2.3, we must eventually obtain $\mathbb{CP}^2$ together with a symplectically embedded 2-sphere of self-intersection number 1 and a collection of symplectically immersed 2-spheres. Since we are assuming that X is a rational homology disk, it follows that we must obtain $\mathbb{CP}^2$ after $k-1$ blow downs and the configuration K must descend to a valid configuration in $\mathbb{CP}^2$. This places strong restrictions on the combinatorial structure of K: all components of K which are disjoint from L must be blown down at some point of this procedure (so in particular they must become (-1)-curves at some earlier point), while a component K_0 of K intersecting L must become a J-holomorphic submanifold of $\mathbb{CP}^2$ of degree $K_0 \cdot L$. This condition, for example, determines the homological square of the image of K_0 in $\mathbb{CP}^2$, and for low degrees it also determines the topology of the result. For most graphs Γ we will reach a homological contradiction at some point of this procedure, showing the nonexistence of the hypothesized rational homology disk filling X.

The graphs that we are not able to rule out with the above strategy correspond precisely to those which are in the lists defining $\mathcal{QHD}_3$ and $\mathcal{QHD}_4$. For these graphs we find certain curve configurations in $\mathbb{CP}^2$, which in turn (after appropriate repeated blow-ups) provide configurations of curves in $\mathbb{CP}^2\#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$ intersecting each other according to the dual graph Γ', and this fact, by the following result of Pinkham, shows that the singularities do admit rational homology disk smoothings.

Theorem 2.12 *([16, Theorem 6.7]) Let Z be a smooth projective rational surface, and $D \subset Z$ a union of smooth rational curves whose intersection graph Γ' is starshaped. Assume*

$$rk\ H_2(D;\mathbb{Z}) = rk\ H_2(Z;\mathbb{Z}).$$

If Γ, the dual of Γ', is the graph of a rational singularity, then one has a rational homology disk smoothing of a rational weighted homogeneous singularity with

resolution graph Γ, and the interior of the Milnor fibre of the smoothing is diffeomorphic to $Z - D$. □

Finally, in proving the equivalences in Theorems 1.4 and 1.6 we will rely on the simple fact that a smoothing of S_Γ is always a symplectic filling (in fact, a Stein filling) of the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ).

3 Small Seifert singularities

By Theorem 2.9 we only need to consider three-legged graphs in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. As is shown in [17] (cf. also Section 5), singularities corresponding to the plumbing trees in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$ do admit rational homology disk smoothings (and therefore the Milnor fillable contact structures admit rational homology disk fillings). Therefore, in determining three-legged graphs with rational homology disk smoothings (or fillings of the corresponding Milnor fillable contact structure) we only need to examine the three-legged graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. The discussion will be given for each of these classes separately; for technical reasons we start with the case of graphs in $\mathcal{C}$.

3.1 Three-legged graphs in the family $\mathcal{C}$

Recall that graphs in $\mathcal{C}$ are defined by repeatedly blowing up the basic configuration shown by Figure 4(c) and then replacing the (-1)-framing with (-2). To get three-legged graphs, we only blow up edges emanating from the (-1)-vertex. There are three cases we distinguish depending on which edge we blow up in the first step in the basic example. The index of the subfamily records the (negative of the) framing of the leaf the first blown up edge points to.

The family $\mathcal{C}_6$

Consider the generic member of the family $\mathcal{C}_6$ depicted in Figure 6(a). As a particular case of Theorem 1.4 we will show

Theorem 3.1 *Suppose that the singularity S_Γ admits resolution graph given by Figure 6(a). Then the following three statements are equivalent:*

(1) *S_Γ admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold* (Y_Γ, ξ_Γ) *admits a weak symplectic rational homology disk filling, and*

(3) *for the graph* Γ *given by Figure 6(a) we have* $b = b_1 = \ldots = b_{n-1} = -2$ *and* $b_n = -n - 5$ *for some positive integer* n.

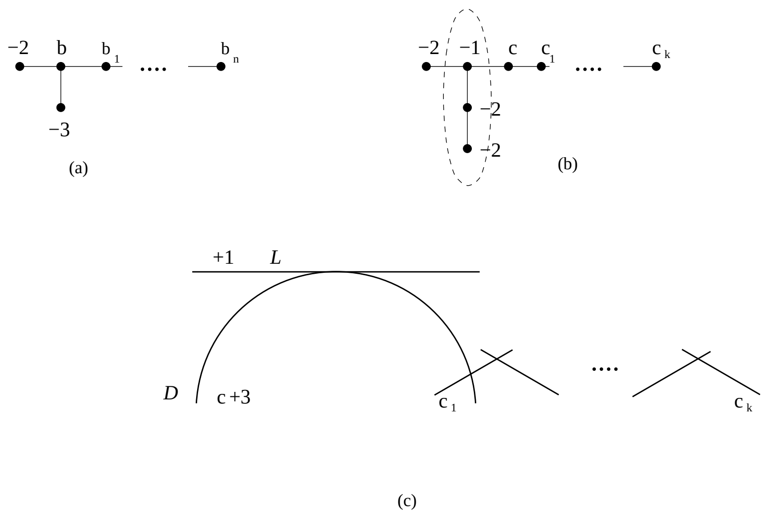

Figure 6: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_6$.

Remark 3.2 Notice that we do not use the full power of Theorem 2.9: although the theorem implies some delicate relation among the coefficients $b_1, \ldots, b_n$ of the graph of Figure 6(a), we will only use the fact that the two other legs are of length one and the framings are -2 and -3. A similar weaker result will be sufficient in all the subcases considered in the present and the subsequent section.

Before turning to the proof of the above result, we start by listing some general observations. The dual graph (after possibly repeatedly blowing up the edge emanating from the central vertex towards the long leg until the central framing becomes -1) has the shape given by Figure 6(b). Blowing down the central vertex together with the two (-2)'s (encircled by the dashed circle in Figure 6(b)), we arrive at the diagram of Figure 6(c); here the curves are symbolized by arcs, and the intersection of two arcs means that the two corresponding curves intersect each other. The resulting $(+1)$-curve will be denoted by L, while the curves of the long leg (with framings $c, c_1, \ldots, c_k$)

will become $D, C_1, \ldots, C_k$, respectively. The tangency between D and L is a triple tangency. Since $b_n \leq -6$, it is easy to see that $k \geq 3$. Notice also that $c_i \leq -2$ once $i \geq 1$ and c is negative. By gluing this compactifying divisor to a potentially existing rational homology disk filling X we get a closed symplectic manifold Z with $b_2(Z) = k+2$. The symplectic 4-manifold Z obviously contains a symplectic $(+1)$-sphere (namely, the curve L), hence it follows by McDuff's Theorem 2.1 that Z is a rational symplectic 4-manifold, that is, a symplectic blow-up of $\mathbb{C}P^2$ at a finite number of points. In particular, Z must be diffeomorphic to $\mathbb{C}P^2 \# (k+1)\overline{\mathbb{C}P^2}$. By repeated applications of Lemma 2.3, we can blow down the pair (Z, L) to obtain $(\mathbb{C}P^2, \text{line})$, while preserving the pseudoholomorphicity of the images of $D, C_1, \ldots, C_k$. Since the curves $C_1, \ldots, C_k$ in the chain are disjoint from the $(+1)$-curve L and are homologically essential, we must blow them down, while the curve D will descend to a cubic curve in $\mathbb{CP}^2$. Since the resulting cubic curve will be the image of a rational curve, it necessarily must contain a singular point. The above observations imply, therefore, that there is a unique additional (-1)-curve E in Z for the chosen almost complex structure, which we have to locate in the diagram. Since J-holomorphic curves intersect positively, the geometric intersections in these cases can be computed via homological arguments.

Proposition 3.3 *Under the above circumstances the exceptional divisor E must intersect the curve D and the curve C_k in the chain in one point each. Consequently, the framings should satisfy $c_i = -2$ for $i = 1, \ldots, k$ and $c = -k+2$.*

Proof Let $\mathcal{J}_K$ denote the nonempty set of tame almost complex structures on Z with respect to which all the curves of $K = L \cup D \cup C_1 \cup \ldots \cup C_k$ are pseudoholomorphic. Choose an almost complex structure J which is generic in $\mathcal{J}_K$. If we blow down all J-holomorphic (-1)-curves away from L, we can show that the chain $C_1, \ldots, C_k$ is transformed into a configuration of curves which can be sequentially blown down. An elementary computation shows that X being a rational homology disk implies that there must be precisely one (-1)-curve E in the complement of L which is not contained in the chain $C_1, \ldots, C_k$. The (-1)-curve E must intersect the chain to start its sequential blow-down. E also must intersect the curve D at least once, since (as D has intersection number 3 with the $(+1)$-curve L) D will become a cubic curve in $\mathbb{CP}^2$. Since the resulting cubic curve is the image of a rational curve, it must admit a singular point, which cannot be achieved by blowing down curves which intersect D at most once. By Corollary 2.5 the curve E cannot intersect the long chain twice. With a similar argument we can see that it can intersect

the chain only in its endpoints: if it intersects the chain in a curve C_i which is not at one of its ends, then blowing down E we get a curve C_i' which now intersects D and two further curves in the chain. When we blow down C_i', the two neighbours will pass through the same point of D. If, now, the image of C_{i-1} is the next curve of the chain to get blown down, then the images of all curves in the portion $C_1, \ldots, C_{i-1}$ of the chain must get blown down before the image of the curve C_{i+1} is blown down. Otherwise, we will get a singular point on the image of D and at least one further curve of the chain passing through through that singular point. After a slight perturbation of the almost complex structure, when (the image) of one of these curves is eventually blown down we will get a further singular point on the image of D, which (with the aid of Lemma 2.7) provides a contradiction. However, after the images of $C_1, \ldots, C_{k-1}$ are blown down, the image of D will become singular, and the same argument again provides a contradiction. If the image of C_{i+1} is the next curve of the chain to get blown down after C_i', then, as before, we can argue that the images of all curves in the portion $C_{i+1}, \ldots, C_k$ of the chain must get blown down before the image of the curve C_{i-1} is blown down. If $i > 3$, then, when the image of C_{i-1} is blown down, we will get a contradiction as before. If $i = 3$, then, when image of C_{i-1} is blown down, we will obtain a singular point on the image of D which has multiplicity greater than 2 and hence its link will not be the trefoil knot or the Hopf link, a contradiction.

If E intersects the chain on its end near D, then after the second blow-down D develops a transverse double point singularity, and the further blow-downs then create more singular points (in the spirit of the argument above), leading to a curve which cannot represent three times the generator in the complex projective plane. Hence the only possibility for the (-1)-curve E is to intersect the chain at its farther end, and intersect D once. In order to blow down all the curves in the chain we must have $c_i = -2$ for $i = 1, \ldots, k$, and since the self-intersection of D will become 9 after all the blow-downs, we derive $c = -k+2$. With this last observation the proof is complete. □

The above lemma then implies that the only possible dual configuration in $\mathcal{C}_6$ which can correspond to a rational homology disk filling is given (with $n = k-3$) by Figure 7(b). After blowing down all the curves disjoint from L in Figure 7(b), we get a configuration consisting of a cubic curve with a transverse double point and a tangent line to it at one of its inflection points; cf. Figure 7(d).

Lemma 3.4 *The configuration of curves given by the graph Γ' of Figure 7(b) does exist in $\mathbb{CP}^2 \# (|\Gamma'| - 1)\overline{\mathbb{CP}^2}$. Consequently the singularity with resolution graph given by Figure 7(a) admits a rational homology disk smoothing.*

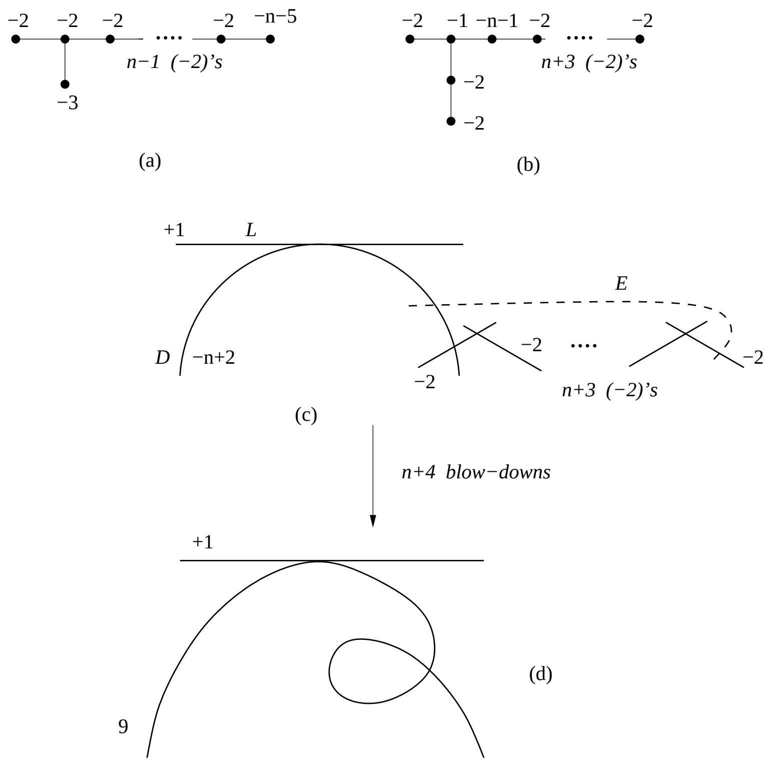

Figure 7: The one-parameter family in $\mathcal{C}_6$ with rational homology disk filling.

Proof Take the singular cubic specified by the degree three homogeneous equation $f(x,y,z) = y^2z - x^3 - x^2z$ in $\mathbb{CP}^2$ and consider the line $\{z = 0\}$ intersecting it in one of its inflection points $[0 : 1 : 0]$. This verifies the existence of the configuration of Figure 7(d) in $\mathbb{CP}^2$. By reversing the blow-down procedure, the existence of the configuration of Figure 7(b) in the appropriate blow-up of $\mathbb{CP}^2$ is proved. By [16, Theorem 6.7] of Pinkham (cf. also Theorem 2.12) the existence of the rational homology disk smoothing of the singularity given by the resolution graph of Figure 7(a) is then verified. □

Proof of Theorem 3.1 Since a smoothing of a singularity always provides a weak filling of the Milnor fillable contact structure of the link, the implication (1) $\Rightarrow$ (2) easily follows. Proposition 3.3 then (after determining Γ from Γ' given in the proposition) provides (2) $\Rightarrow$ (3). Finally, Lemma 3.4 implies (3) $\Rightarrow$ (1), concluding the proof. □

Remarks 3.5 **(a)** The scheme of the proof of the other cases for the three-

and four-legged graphs in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ will be very similar, although the ad hoc arguments given in Proposition 3.3 will significantly vary.

(b) Notice that we did not use Theorem 2.9 in its full power; we only needed that the graphs we are examining have two legs of length one, on which the framings are (-2) and (-3). Again, this will be a recurring theme.

(c) In the nonexistence argument we only used homological considerations regarding self-intersections and intersection numbers, with the only exception regarding smoothness of the curves to be blown down and the singularity of the resulting cubic curve.

(d) The number of additional (-1)-curves we had to locate was dictated by the fact that the filling is a rational homology disk. For fillings with richer homology theory, a similar method applies, although the combinatorial argument will get more involved as the number of (-1)-curves increases. In our subsequent discussions we will meet examples where two or three such curves are needed to be located.

(e) The existence of the curve configuration in $\mathbb{CP}^2$ is a truly geometric problem, which admits a very simple solution in this case, and can be rather complicated for other cases; cf. Lemma 4.11, for example.

(f) It is fairly straightforward to see that the family of graphs within $\mathcal{C}_6$ for which the rational homology disk smoothing exists is given by the defining procedure of $\mathcal{C}$ when we always blow up the edge emanating from the (-1)-vertex which connects it with the leaf.

(g) Note that all the configurations appearing in Theorem 3.1(3) are, in fact, members of the $\mathcal{M}$ family: take $q = n - 1$ in Figure 1(f).

The family $\mathcal{C}_3$

The generic member of this family is given by Figure 8(a), together with the dual graph and the result of the triple blow-down.

Theorem 3.6 *Suppose that the singularity S_Γ admits resolution graph given by Figure 8(a). Then the following three statements are equivalent:*

(1) *S_Γ admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

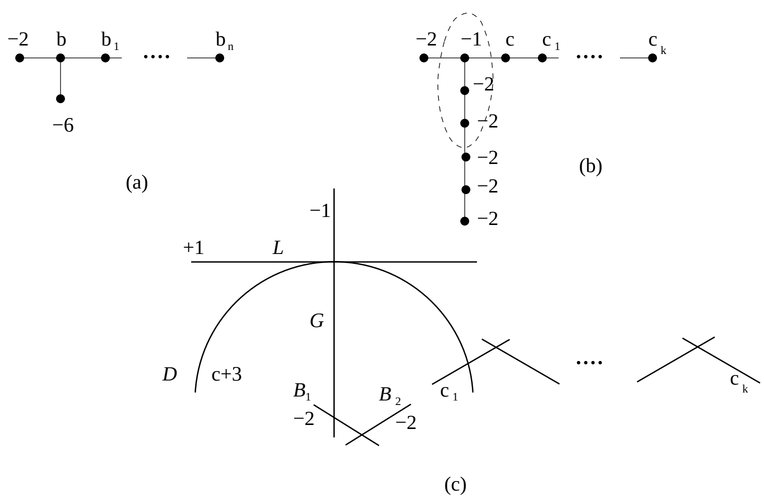

Figure 8: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_3$.

(3) *for the graph* Γ *given by Figure 8(a) either* $b = b_1 = \ldots = b_{n-1} = -2$ *and* $b_n = -n-2$ *for some positive integer* n, *or* $b = b_1 = \ldots = b_{n-4} = b_{n-2} = b_{n-1} = -2$, $b_{n-3} = -3$ *and* $b_n = -n-1$ *for some positive integer* $n \geq 4$.

By gluing the compactifying divisor given by Figure 8(c) to a potentially existing rational homology disk filling X we get a closed symplectic manifold Z, and a simple count shows that $b_2(Z) = k+5$. The symplectic 4-manifold Z obviously contains a symplectic $(+1)$-sphere (namely, the curve L), hence, by McDuff's Theorem 2.1, Z is diffeomorphic to $\mathbb{C}P^2 \# (k+4)\overline{\mathbb{C}P^2}$. By repeated applications of Lemma 2.3, we can blow down the pair (Z, L) to obtain $(\mathbb{C}P^2, \text{line})$, while preserving the pseudoholomorphicity of the images of $D, C_1, \ldots, C_k, B_1, B_2$. Since the curves $C_1, \ldots, C_k$ and B_1, B_2 are disjoint from the $(+1)$-curve L and are homologically essential, we must blow them down. This means that there are two further (-1)-curves E_1 and E_2 which we have to locate in the diagram. For a generic almost complex structure these curves will be disjoint (-1)-curves. Since both B_1 and B_2 have to be blown down (being disjoint from the $(+1)$-curve L), one of them must intersect one of the (-1)-curves, say E_1. Since the complement of the $(+1)$-curve does not contain homologically essential spheres with nonnegative square, E_2 then cannot intersect any of the B_i.

Proposition 3.7 *Under the above circumstances, the existence of a rational homology disk smoothing X implies that E_2 intersects D and C_k, and E_1 either intersects B_1 and D or B_2 and C_1. The self-intersections in these two cases are $c = -k-1$ and $c_1 = \ldots = c_k = -2$ or $c_1 = -k+2$, $c_1 = -5$ and $c_2 = \ldots = c_k = -2$.*

Proof Case I: Suppose that $E_1 \cdot B_1 > 0$. After three blow-downs the curve G becomes a $(+1)$-curve, so it cannot be blown down any further: in $\mathbb{CP}^2$ it will be a curve intersecting the $(+1)$-curve once, hence it will be a line with self-intersection $+1$. Therefore, to prevent further blow-downs along the points of the vertical curve, $E_2 \cdot G = 0$ and E_1 must be disjoint from the long chain. So E_2 must intersect the long chain, and since the whole chain must be blown down, a simple adaptation of Proposition 3.3 gives that the only possibility for E_2 is the one given by Figure 9(c). Notice that the images of G and D must intersect each other three times after all curves have been blown down, which can be achieved only if E_1 intersects D exactly once. (Recall that E_2 must stay disjoint from G.) This argument shows that the only possibility for E_1 and E_2 (under the assumption $E_1 \cdot B_1 > 0$) is given by the dashed lines of Figure 9(c). The blowing down process then dictates the values of c and c_i, leading us (with $k = n$) to the configuration given by Figure 9.

Case II: Suppose now that $E_1 \cdot B_2 > 0$. Then after three blow-downs the vertical curve G becomes a 0-curve, so either (a) E_2 intersects G or (b) E_1 intersects a further (-1)-curve in the chain (after it has been partially blown down). If E_1 intersects B_2 and E_2 intersects G then none of the E_i intersect the chain, and since the chain is nonempty, this provides a contradiction.

Therefore E_1 should intersect the long chain, and it should intersect it in the last curve to be blown down from there. Suppose that $E_1 \cdot C_i = 1$. Then E_1 cannot intersect D, since otherwise after blowing down E_1, then sequentially blowing down the images of B_2 and B_1, C_i', the image of C_i, will intersect the image of D at least three times (counting with multiplicity). When (the image of) C_i' is eventually blown down the image of D will gain a singularity which is not permitted for a cubic in $\mathbb{C}P^2$. This shows that E_2 has to intersect the chain (and start the sequence of blow-downs) and it also has to intersect D to get a singularity on it. Furthermore, we also know that E_2 must be disjoint from G. The argument of Proposition 3.3 shows that E_2 must intersect the long chain at its farther end and also D, as depicted (with $k = n$) in Figure 10. As usual, the framings are dictated by the fact that all curves in the complement of the $(+1)$-curve must be blown down. $\square$

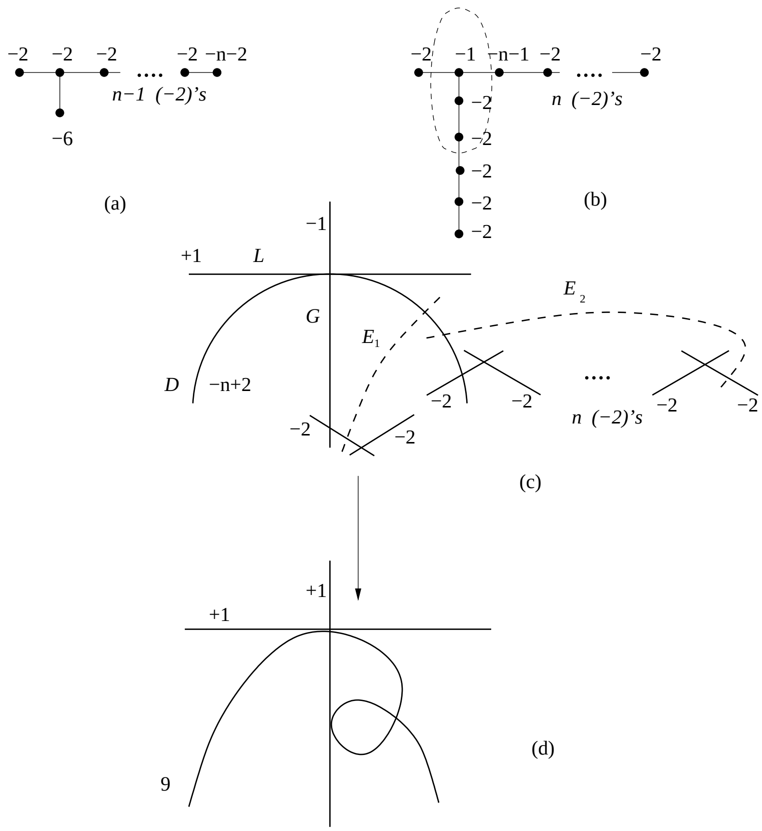

Figure 9: A one-parameter family in $\mathcal{C}_3$ with rational homology disk filling.

Lemma 3.8 *The configuration of curves given by the graph Γ' either of Figure 9(b) or of 10(b) do exist in $\mathbb{CP}^2\#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$. Consequently the singularities with resolution graphs given by Figure 9(a) or 10(a) do admit a rational homology disk smoothings.*

Proof For Case I of Proposition 3.7 above the required configuration clearly exists: consider a cubic with a transverse double point and its tangent at one of its inflection point as in the proof of Lemma 3.4, and add a further line through the inflection point which is transverse there, but is tangent to the cubic at a further point. The singular cubic given by equation $f(x,y,z)=y^2z-x^3-x^2z$ in $\mathbb{CP}^2$ and the line $\{z=0\}$ together with $\{x+z=0\}$ (intersecting the cubic in the inflection point $[0:1:0]$ and being tangent to it at $[-1:0:1]$), for example, is such a choice.

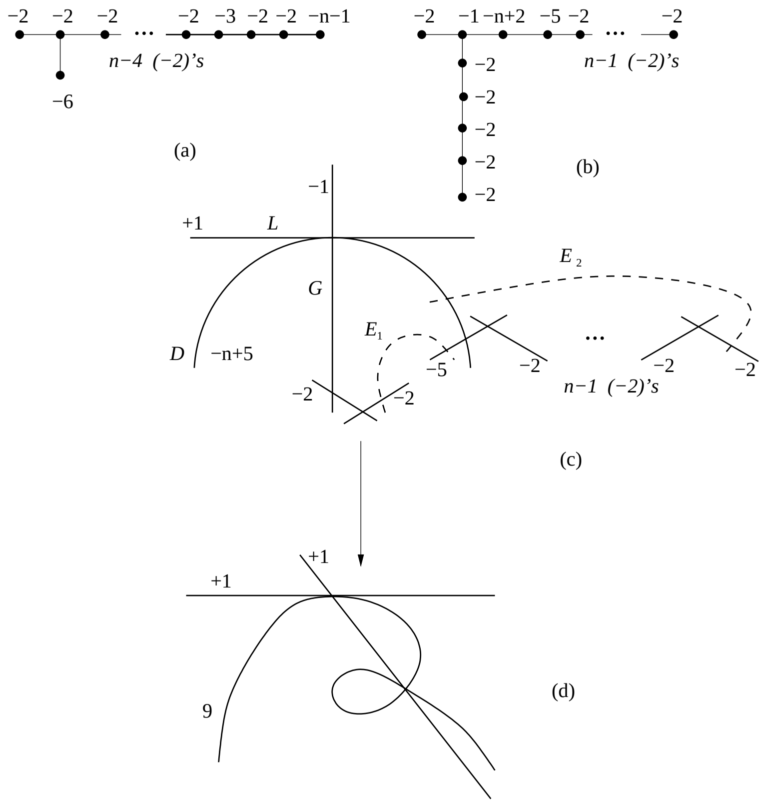

Figure 10: A further one-parameter family in $\mathcal{C}_3$ with rational homology disk filling.

In Case II of Proposition 3.7 the configuration of the cubic and the two lines (one tangent at an inflection point, the other passing through the inflection point and the transverse double point) clearly exists: take the two curves as in the proof of Lemma 3.4 and add $\{x = 0\}$ (the line passing through the inflection point $[0 : 1 : 0]$ and the transverse double point $[0 : 0 : 1]$ of the cubic).

These configurations (after the appropriate blow-ups) embed the dual graphs in the appropriate rational surfaces, hence Pinkham's result Theorem 2.12 shows that the rational homology disk smoothings exist. Case I above has been already treated (by slightly different means) in [17, Example 8.6]. $\square$

Proof of Theorem 3.6 As in the proof of Theorem 3.1, the implication (1) $\Rightarrow$ (2) follows from the general principle that a smoothing of a singularity always

provides a weak filling of the Milnor fillable contact structure on the link. After determining Γ from its dual graph Γ', Proposition 3.7 provides (2) $\Rightarrow$ (3), while Lemma 3.8 implies (3) $\Rightarrow$ (1), concluding the proof. $\square$

Remark 3.9 Once again, we get the first family described in Theorem 3.6 by starting with the graph defining the family $\mathcal{C}$ and always blowing up the edge from the (-1)-vertex which connects it with the leaf. The construction of the second family of Theorem 3.6 is slightly unusual: in the blow-up procedure creating elements of $\mathcal{C}_3$ we always blow-up the edge connecting the (-1)-vertex with the leaf, except in the last blow-up, when we blow up the *other edge* emanating from the (-1)-vertex. After this unusual move we substitute the (-1)-framing with (-2) and arrive at the graphs depicted by Figure 10. Notice that this graph already appears in the family $\mathcal{M}$; compare with Figure 1(d) with $q = n-4$ and $r = 3$.

The family $\mathcal{C}_2$

The generic case in this family is shown by Figure 11(a).

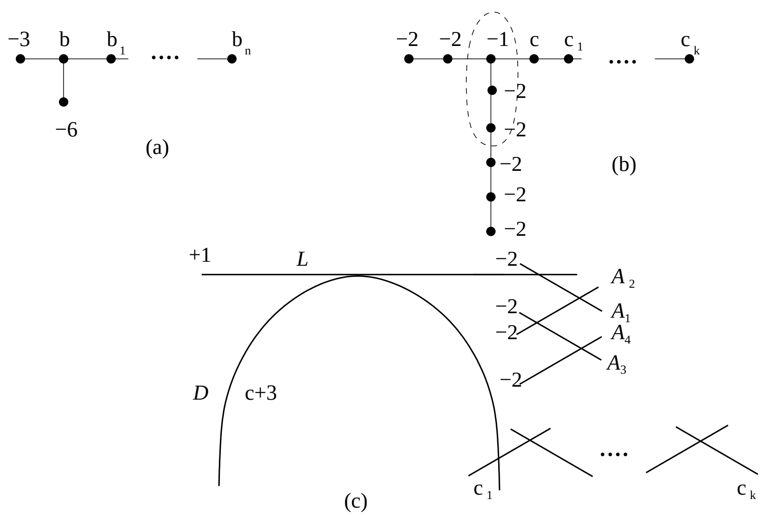

Figure 11: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_2$.

Theorem 3.10 *Suppose that the singularity S_Γ admits resolution graph given by Figure 11(a). Then the following three statements are equivalent:*

(1) S_Γ *admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold* (Y_Γ, ξ_Γ) *admits a weak symplectic rational homology disk filling, and*

(3) *for the graph* Γ *given by Figure 11(a) either* $b = b_1 = \ldots = b_{n-1} = -2$ *and* $b_n = -n-1$ *for some positive integer* n, *or* $b = b_1 = \ldots = b_{n-5} = b_{n-3} = b_{n-2} = -2$, $b_{n-4} = b_{n-1} = -3$ *and* $b_n = -n+1$ *for some* $n \geq 5$ *or* $b = b_1 = \ldots = b_{n-5} = b_{n-3} = b_{n-2} = b_{n-1} = -2$, $b_{n-4} = -4$ *and* $b_n = -n+1$ *for some* $n \geq 5$.

The usual simple calculation shows that by assuming the existence of a rational homology disk filling for (Y_Γ, ξ_Γ) we have to locate two (-1)-curves in the diagram, which we will call E_1 and E_2. Since the curves A_2, A_3 and A_4 must be blown down at some point in the blow-down procedure, one of the (-1)-curves (say E_1) should intersect $A_2 \cup A_3 \cup A_4$.

Proposition 3.11 *In the situation under examination, the existence of a rational homology disk filling implies that* E_2 *intersects* D *and* C_k, *while* E_1 *either intersects* A_2 *and* D *or* A_4 *and* C_2 *or* A_4 *and* C_3. *The framings in the three cases are given by* $c = -k-2$ *and* $c_1 = \ldots c_k = -2$, *or* $c_1 = -k+3$, $c_2 = -5$, $c_3 = -3$ *and* $c_4 = \ldots = c_k = -2$, *or* $c = -k+2$, $c_2 = -6$ *and* $c_1 = c_3 = \ldots c_k = -2$.

Proof Notice first that E_1 cannot intersect A_3 (otherwise we will have a self-intersection 0 curve in the complement of L, contradicting Lemma 2.6); hence we have two cases to examine.

Case I: Suppose that E_1 intersects A_2, i.e., $E_1 \cdot A_2 > 0$. In this case, after four blow-downs, the self-intersection of A_1 becomes 1, which cannot go any higher, since in $\mathbb{CP}^2$ the curve A_1 will become a line. Therefore E_1 must be disjoint from the chain and E_2 must be disjoint from all the A_i's. In order for the image of A_1 to intersect D three times, E_1 must intersect D. Since E_2 is disjoint from all the A_i's, and it starts the blow-down of the chain, and is responsible for the singularity on D, the usual argument presented in the proof of Proposition 3.3 locates it. In conclusion, the only possibility is shown (with $n = k+1$) in Figure 12, together with the framings dictated by this one-parameter family. After repeatedly blowing down, we arrive at a curve configuration involving a singular cubic with two tangent lines at its two inflection points.

Case II: Suppose now that E_1 intersects A_4. After blowing down E_1, and then sequentially blowing down the images of A_4, A_3 and A_2, the self-intersection

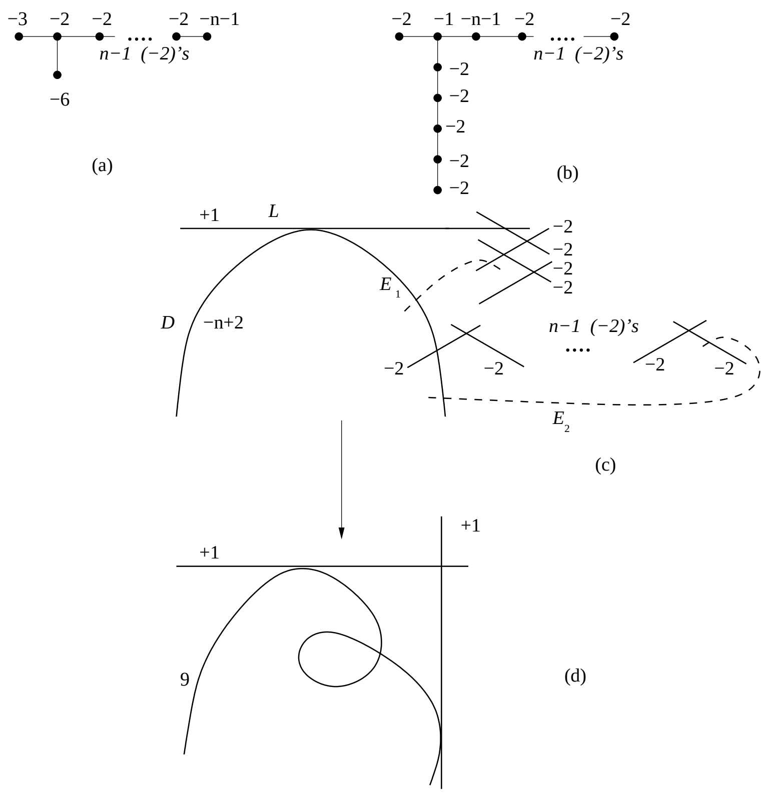

Figure 12: A one-parameter family in $\mathcal{C}_2$ with rational homology disk filling.

of A_1 will increase to (-1). In order to increase it to $(+1)$ we have a number of possibilities.

(i) $E_1 \cdot C_i = 0$, i.e., E_1 is disjoint from the chain. In this case E_2 must intersect A_1 and also the last curve we blow down in the chain. Since then there is no further curve starting the blow-down of the chain, this can happen only if the chain has a single element. If E_2 is disjoint from D, then after all blow-downs have been carried out D remains smooth, which is a contradiction. Therefore E_2 must intersect D. Blowing down E_2 and then the elements in the chain we get that the image of A_1 passes through D three times. Therefore E_1 must be disjoint from D. Computing the self-intersections, however, we see that the curve with framing c (giving rise to D, which will become of self-intersection 9) must have self-intersection $c = +1$ in the dual graph, which is a contradiction.

(ii) Assume now that E_1 intersects the chain in the curve we will blow down

last. This implies that E_2 should intersect A_1, but since the blow-down of E_1 (together with the last curve in the chain) increases the self-intersection of A_1 by two, E_2 must be disjoint from the chain. Therefore, once again, the chain must be of length one. Performing the blow-downs, we conclude that D remains smooth and the images of D and A_1 will intersect each other only twice, hence this case does not occur.

(iii) Finally, it can happen that E_1 intersects the chain in the penultimate curve to get blown down. Then E_2 should be disjoint from the A_i's, and since the singularity on D cannot be caused by blowing down E_1, we need that E_2 intersects D. The usual argument given in the proof of Proposition 3.3 shows the position of E_2, leading to two configurations, depending on whether the last curve to be blown down is next to D or is one off. The resulting possibilities (with $n = k + 1$) are given by Figures 13 and 14. ◻

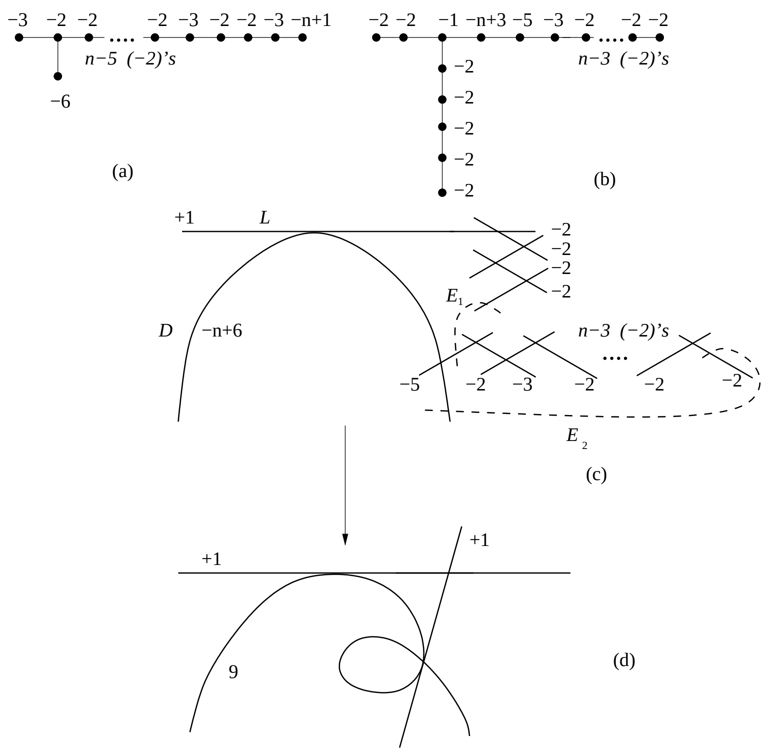

Figure 13: A further one-parameter family in $\mathcal{C}_2$ with rational homology disk filling.

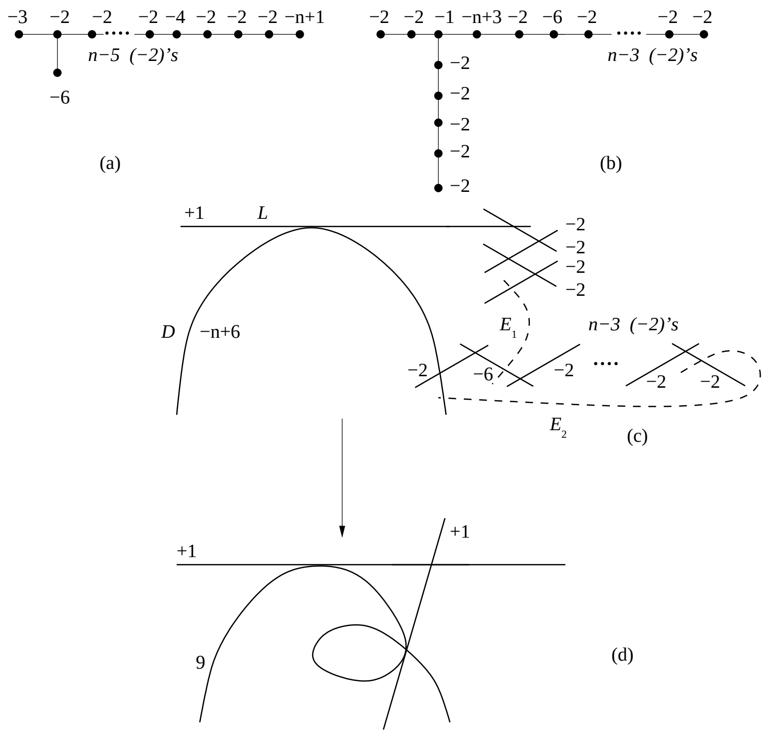

Figure 14: One more one-parameter family in $\mathcal{C}_2$ with rational homology disk filling.

Lemma 3.12 *The configuration of curves given by the graph Γ' either of Figure 12(b), of 13(b) or of 14(b) does exist in $\mathbb{CP}^2\#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$. Consequently the singularities with resolution graphs given by Figure 12(a), 13(a) or 14(a) do admit rational homology disk smoothings.*

Proof In Case I of Proposition 3.11 consider the cubic and one tangent already studied in the the proof of Lemma 3.4, together with the tangent line $\{y-(x+\frac{8}{9}z)\sqrt{3}i = 0\}$ passing through another inflection point $[-\frac{4}{3} : -i\frac{4}{3\sqrt{3}} : 1]$ of $y^2z - x^3 - x^2z$. (It is not hard to see that the further two inflection points of the curve $y^2z - x^3 - x^2z$ are $[-\frac{4}{3} : \pm i\frac{4}{3\sqrt{3}} : 1]$.)

In Case II of Proposition 3.11 we need a nodal cubic, with a tangent at one of its inflection points and a tangent to one of its branches at its nodal point. The cubic and the tangent at the inflection point can be chosen as in the proof of Lemma 3.4, and the additional tangent can be chosen to be $\{x = y\}$ or

$\{x = -y\}$.

Having these curves in $\mathbb{CP}^2$, the rest of the proof is identical to the previous cases, e.g. in Lemma 3.4: the appropriate blow-ups embed the dual graphs in the right blow-ups of $\mathbb{CP}^2$ and then an application of Pinkham's Theorem 2.12 completes the argument. $\square$

Proof of Theorem 3.10 Once again, the implication $(1) \Rightarrow (2)$ follows from the general principle that a smoothing of a singularity always provides a weak filling of the Milnor fillable contact structure on the link. After converting the dual graph Γ' to Γ, Proposition 3.11 provides $(2) \Rightarrow (3)$, while Lemma 3.12 implies $(3) \Rightarrow (1)$, concluding the proof. $\square$

Remark 3.13 As before, the graphs found in Case I of Proposition 3.11 are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf. The graphs in Case II are constructed in a slightly unusual manner: In constructing the plumbing graph for the first case (depicted by Figure 13) we blow up the edge emanating from the (-1)-vertex pointing to the leaf, except in the penultimate blow-up, where we choose the other edge, but for the last blow-up we choose the edge connecting the (-1)-vertex with the neighbour of the leaf. In the second case the graph is constructed by repeatedly blowing up the edge connecting the (-1)-vertex with the leaf, and then in the penultimate step we blow up the other edge, and finally we blow up the edge which is *not* connecting the (-1)-vertex to the neighbour of the leaf. The resulting two one-parameter families are given by the figures. Notice again, that these graphs already appeared in our previous lists as members of the family $\mathcal{M}$; compare with Figure 1(g) with $p = 1$, $r = 3$ and $q = n - 5$, and Figure 1(e) with $p = 4$ and $q = n - 5$.

3.2 Graphs in $\mathcal{A}$

For three-legged graphs in $\mathcal{A}$ there is no need for further subdivisions since the legs in this case are symmetric. As usual, the generic member of the family is shown by Figure 15(a).

Theorem 3.14 *Suppose that the singularity S_Γ admits resolution graph given by Figure 15(a). Then the following three statements are equivalent:*

1. *S_Γ admits a rational homology disk smoothing,*
2. *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

(3) *for the graph* Γ *given by Figure 15(a)* $b = b_1 = \ldots = b_{n-2} = -2$, $b_{n-1} = -4$ *and* $b_n = -n-2$ *for some positive integer* $n \geq 2$.

The usual simple count shows that if we assume the existence of a rational homology disk filling, then we have to find two (-1)-curves E_1, E_2; cf. Figure 15. The curve A is of self-intersection (-1), and will become a line in $\mathbb{CP}^2$, hence must be hit by one of the (-1)-curves, say by E_1.

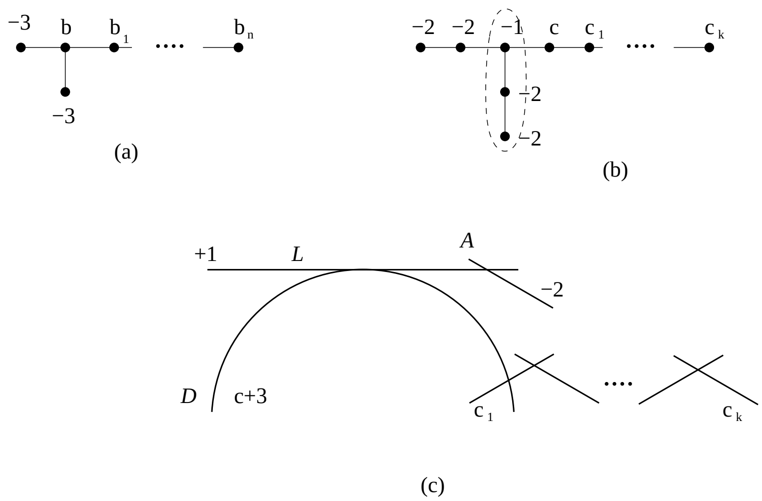

Figure 15: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{A}$.

Proposition 3.15 *In this case, the curve* E_2 *intersects* D *and* C_k, *while* E_1 *intersects either* A *and* C_1 *or* A *and* C_2. *The corresponding framings in both cases are* $c = -k+2$, $c_2 = -3$ *and* $c_1 = c_3 = \ldots c_k = -2$.

Proof We are assuming that E_1 intersects A. If E_2 also intersects A, then only one of them (say E_2) can intersect the long chain, and only in the last curve to be blown down, so we cannot start the blow-down process on the chain unless it is of length one. We show that this case never occurs. In fact, to create the singularity on D, the (-1)-curve E_2 must intersect it, and so by blowing down E_2 and the unique element in the chain, we get that the resulting A and D will intersect each other three times, hence E_1 must be disjoint from D. The self-intersection of the resulting singular cubic (which must be equal to 9) is $c+8$, implying that $c = 1$, which contradicts the fact that it should be negative.

Therefore E_2 cannot intersect A, and so it must intersect the long chain, and to create the singular point on D it must also intersect that curve. The usual argument already discussed in Proposition 3.3 shows that E_2 can intersect the chain only in C_k. In order to raise the self-intersection of A from (-2) to 1 we need that E_1 intersect the chain in the penultimate curve to be blown down. Since after the blow-downs the image of A will pass through the singular point of D, E_1 must be disjoint from D. The two very similar possibilities for the (-1)-curves (differing only in the position of the E_1-curve) are shown (with $k = n + 2$) by Figures 16(c) and (d), where also the one-parameter family of framings are indicated. $\square$

Lemma 3.16 *The configuration of curves given by the graph Γ' of Figure 16(b) does exist in $\mathbb{CP}^2 \#(|\Gamma'| - 1)\overline{\mathbb{CP}^2}$. Consequently the singularities with resolution graphs given by Figure 16(a) do admit rational homology disk smoothings.*

Proof By adding $\{x = y\}$ (or $\{x = -y\}$) to the two curves we chose in the proof of Lemma 3.4 we get the configuration of curves in $\mathbb{CP}^2$ depicted in Figure 16(e). The appropriate blow-up sequence then shows that the dual configuration Γ' embeds in an appropriate rational surface, hence an application of Pinkham's Theorem 2.12 concludes the proof. $\square$

Proof of Theorem 3.14 As usual, the implication $(1) \Rightarrow (2)$ follows from general principles, while Proposition 3.15 (after converting the dual graph back to Γ) provides $(2) \Rightarrow (3)$. Finally, Lemma 3.16 implies $(3) \Rightarrow (1)$. $\square$

Remark 3.17 The graphs found in this case are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf. We also note here that all graphs given in Theorem 3.14(3) already appeared in the family $\mathcal{M}$: consider Figure 1(e) with $p = 1$ and $q = n - 2$.

3.3 Graphs in $\mathcal{B}$

Similarly to the case of $\mathcal{C}$, the study of the family $\mathcal{B}$ falls into two subcases, depending on the choice of the first blow-up.

The family $\mathcal{B}_4$

The generic member of this family (together with the dual graph and the configuration of curves after three blow-downs) is shown in Figure 17.

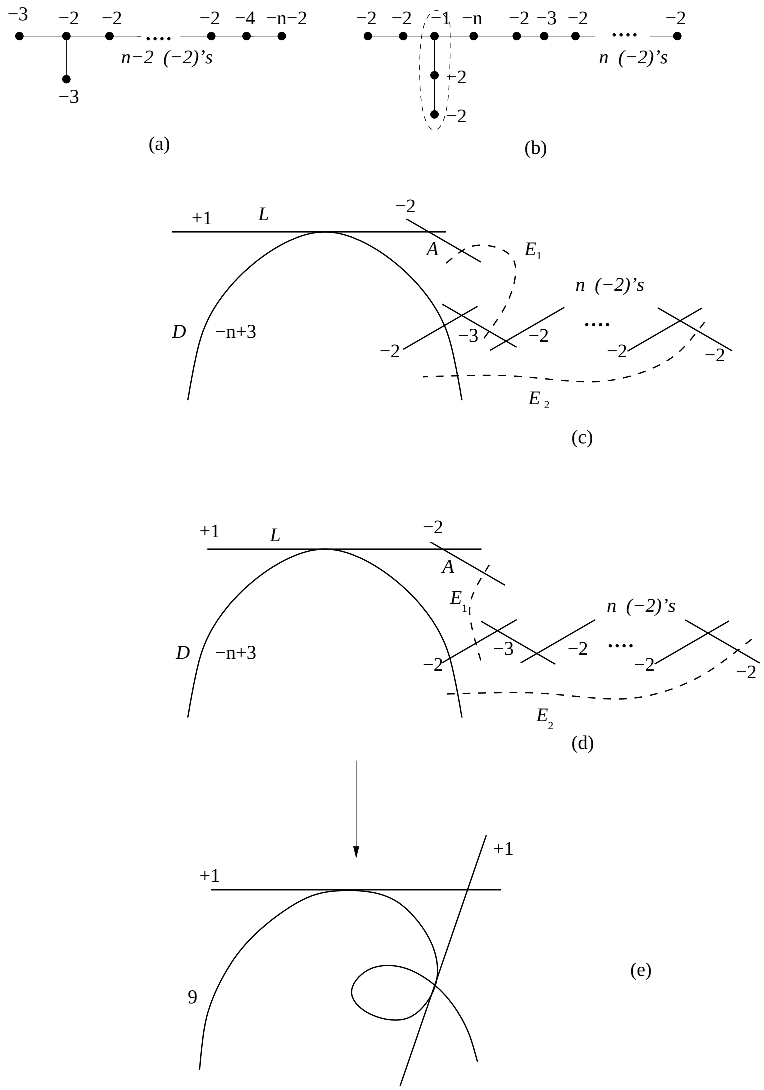

Figure 16: The one-parameter family in $\mathcal{A}$ with rational homology disk filling.

Theorem 3.18 *Suppose that the singularity S_Γ admits resolution graph given by Figure 17(a). Then the following three statements are equivalent:*

(1) *S_Γ admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

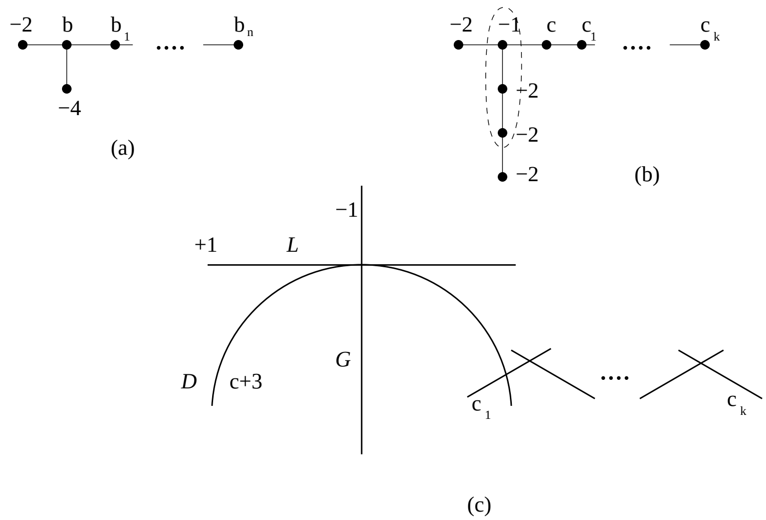

Figure 17: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{B}_4$.

(3) *for the graph* Γ *given by Figure 17(a)* $b = b_1 = \ldots = b_{n-2} = -2$, $b_{n-1} = -3$ *and* $b_n = -n-3$ *for some positive integer* $n \geq 2$.

The usual count of curves shows that we need to locate two (-1)-curves in order to show the existence of a rational homology disk filling. As usual, these two curves will be denoted by E_1 and E_2. It is clear that one of them, say E_1, must intersect G in order to increase its self-intersection to 1.

Proposition 3.19 *Under the above hypotheses, the existence of a rational homology disk filling implies that* E_2 *intersects* D *and* C_k, *while* E_1 *intersects* G *and* C_1. *The corresponding framings are* $c = -k+2$, $c_1 = -3$ *and* $c_2 = \ldots c_k = -2$.

Proof If E_2 also intersects G then both E_1 and E_2 must be disjoint from the chain, hence it cannot be blown down. Therefore we can assume that E_2 is disjoint from G, and therefore E_1 must intersect the chain in the last curve to be blown down. The curve E_1 must be disjoint from D, since if E_1 intersects D then after two blow-downs the curves resulting from G and D will intersect at least four times, giving a contradiction. Therefore E_1 must be disjoint from D, hence E_2 intersects the configuration of curves as is found in the proof of Proposition 3.3. The only possibility is then shown (with $k = n+2$) by Figure 18, together with the possible framings. $\square$

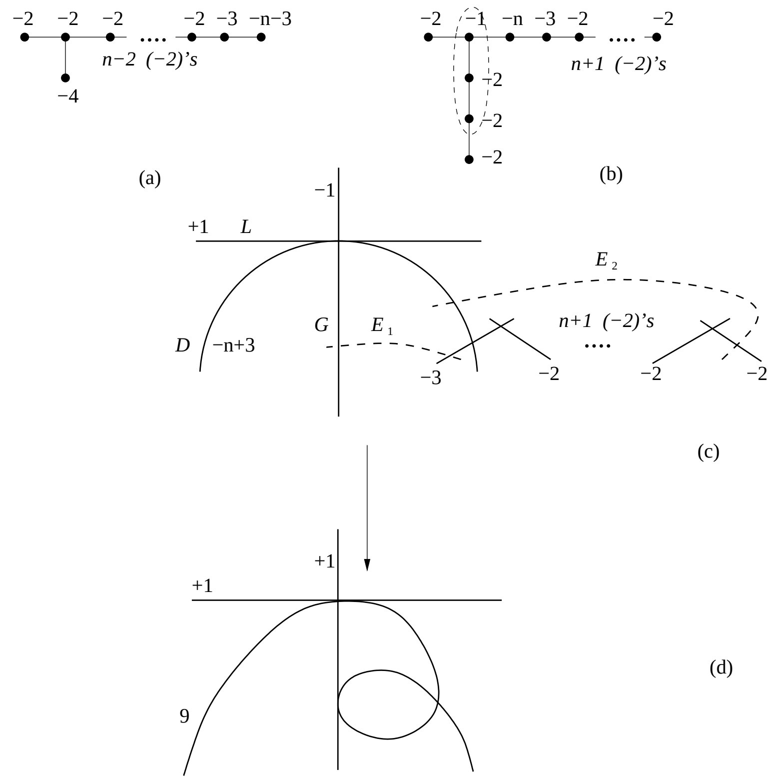

Figure 18: The one-parameter family in $\mathcal{B}_4$ with rational homology disk filling.

Lemma 3.20 *The configuration of curves given by the graph Γ' of Figure 18(b) does exist in $\mathbb{CP}^2\#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$. Consequently the singularities with resolution graphs given by Figure 18(a) do admit rational homology disk smoothings.*

Proof The configuration of curves shown by Figure 18(d) obviously exists in $\mathbb{CP}^2$: choose the cubic and the tangent line (at an inflection point) as in the proof of Lemma 3.4, and add the line $\{x=0\}$ to them. Blowing this configuration up, we arrive at an embedding of the dual curve configuration, which by Pinkham's Theorem 2.12 implies the existence of a rational homology disk smoothing. $\square$

Proof of Theorem 3.18 As usual, the implication $(1)\Rightarrow(2)$ follows from general principles, while Proposition 3.19 provides $(2)\Rightarrow(3)$. Finally, Lemma 3.20 implies $(3)\Rightarrow(1)$. $\square$

Remark 3.21 The graphs found in this case are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf.

The family $\mathcal{B}_2$

The graphs (with their duals, and the curve configuration we get by the three blow-downs) are shown in Figure 19.

Theorem 3.22 *Suppose that the singularity S_Γ admits resolution graph given by Figure 19(a). Then the following three statements are equivalent:*

(1) *S_Γ admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

(3) *for the graph Γ given by Figure 19(a) $b = b_1 = \ldots = b_{n-2} = -2$, $b_{n-1} = -3$ and $b_n = -n-1$ for some positive integer $n \geq$ or $b = b_1 = \ldots = b_{n-4} = b_{n-1} = -2$, $b_{n-2} = b_{n-3} = -3$, $b_n = -n$ for some integer $n \geq 4$.*

The usual curve count shows that for identifying a rational homology disk filling we must find three (-1)-curves E_1, E_2, E_3 in the diagram. Suppose that E_1 intersects G.

Proposition 3.23 *Under the circumstance described above, from the existence of a rational homology disk filling it follows that either*

- *the curve E_3 intersects D and C_k, E_2 intersects C_1 and A_1 and E_1 intersects G, D and A_2 and therefore the framings satisfy $c = -k$, $c_1 = -3$ and $c_2 = \ldots = c_k = -2$, or*
- *E_3 intersects D and C_k, E_2 intersects A_2 and C_2, and E_1 intersects G and C_1 and therefore the framings are given as $c = -k+2$, $c_1 = -3$, $c_2 = -4$, $c_3 = \ldots = c_k = -2$, or*
- *E_3 intersects D and C_k, E_2 intersects A_2 and C_1, and E_1 intersects G and C_2 and the framings are $c = -k+2$, $c_1 = -3$, $c_2 = -4$, $c_3 = \ldots = c_k = -2$.*

Proof Since G has self-intersection (-1) and it intersects the curve L once, its self-intersection must increase to 1, hence either another (-1)-curve, say

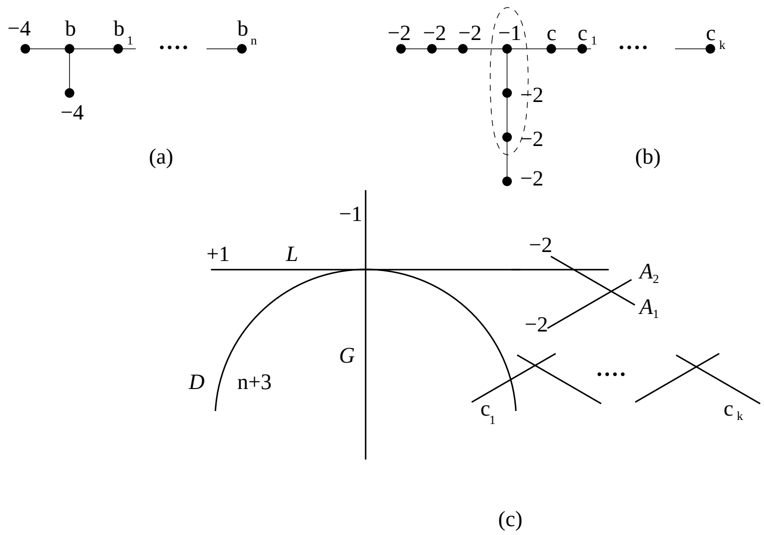

Figure 19: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{B}_2$.

E_2, intersects G or E_1 intersects either A_2 or the chain. Note that E_1 cannot intersect both A_2 and the chain, since if $E_1 \cdot C_i = 1$ and $E_2 \cdot A = 1$, then after E_1 and the image of A_2 are blown down the image of C_i will become tangent to the image of G. When the image of C_i is eventually blown down, the image of G will gain a singularity, which is impossible for a line in $\mathbb{C}P^2$.

Case I: $E_2 \cdot G = 1$. In this case both E_1 and E_2 must be disjoint from A_2 and the chain, hence E_3 intersects both A_2 and the chain. Also, since G and A_1 will intersect after the blowing down process has been carried out, E_1 or E_2 (say E_1) must intersect A_1. After blowing down the E_i's and the image of A_2, the self-intersection of A_1 will already be zero, hence E_3 can only intersect the chain in the last curve to get blown down, which is possible only if the chain is of length one. If E_3 is disjoint from D then (in order for A_1 to intersect D three times) E_1 must intersect D twice, and hence (in order to avoid $G \cdot D > 3$) the curve E_2 must be disjoint from D. Now we can easily see that the self-intersection of D increases to $c + 8$ after all the blow-downs have been performed, and since it should be equal to 9, we deduce that $c = 1$, contradicting the fact that c_1 is negative. If E_3 intersects D then after blowing down E_3 and then sequentially blowing down the images of A_2 and the unique element in the chain we get a singularity on D of multiplicity 3, a contradiction. This shows that Case I, in fact, cannot occur.

Case II: $E_1 \cdot A_2 = 1$. Then both E_2 and E_3 must be disjoint from G, and one of them (say E_2) intersects A_1. To increase the self-intersection of A_1, the curve E_2 should intersect the chain in the last curve to be blown down. Since the image of G will intersect D, we see that $E_1 \cdot D = 1$. This implies that after blowing down E_1 and A_2, the curve A_1 will intersect D once, therefore E_2 cannot intersect D (since it would add three to $A_1 \cdot D$). Now the usual argument from the proof of Proposition 3.3 shows that E_3 starts the blow-down of the chain, and it also intersects D in one point, leading to the configuration depicted (with $n = k$) in Figure 20, where also the necessary framings are indicated.

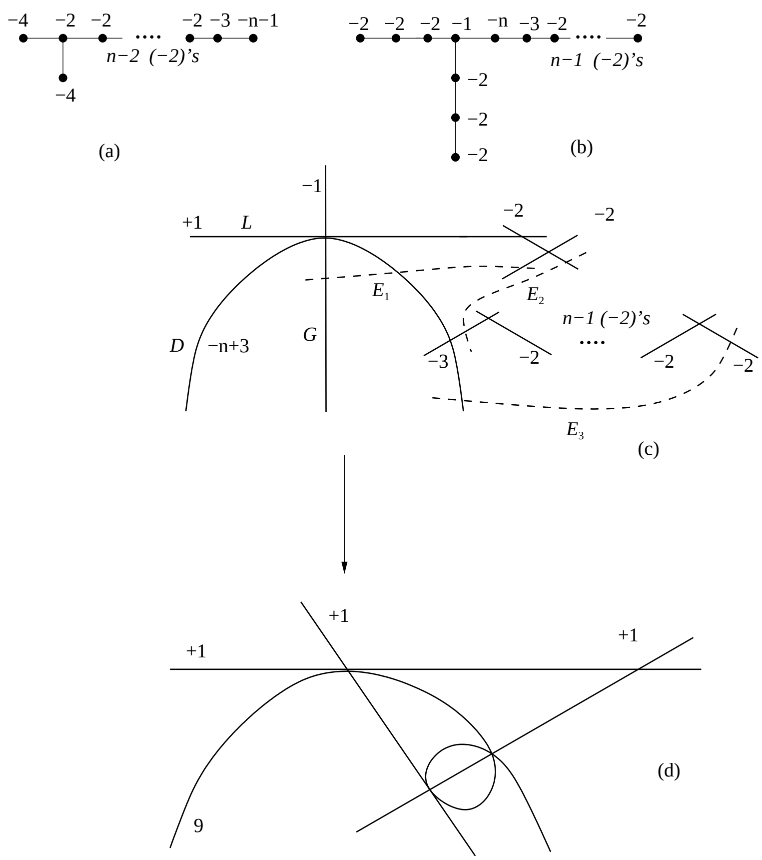

Figure 20: The one-parameter family in $\mathcal{B}_2$ with rational homology disk filling.

Case III: $E_1 \cdot C_i = 1$. Recall that by the previous argument we can assume that $E_2 \cdot G = E_3 \cdot G = 0$. If E_2 and E_3 are both disjoint from the chain, then

the chain must have length one. But then, if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of G and D will be less than 3 and if $E_1 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of G and D will be greater than 3, both contradicting the fact that the intersection number of a line and a cubic in $\mathbb{CP}^2$ is equal to three. So we may assume that E_3 intersects the chain, say $E_3 \cdot C_l = 1$, and, by the preceding argument, that $E_1 \cdot D = 0$. If $E_3 \cdot D = 0$, again we find that, after the blowing down process has been carried out, the intersection number of the images of G and D will be 2, a contradiction. So we must have $E_3 \cdot D = 1$. Now observe that we must have $E_3 \cdot A_2 = 0$. Indeed, if $E_3 \cdot C_j = 1$ and $E_3 \cdot A_2 = 1$, then after E_3 and the image of A_2 are blown down, the image C_j' of C_j will be tangent to the image D. It is now easy to see that after the blowing down process is complete the image of D will have more than one singular point or a singularity of multiplicity greater that 2, both of which are impossible for a cubic in $\mathbb{CP}^2$. Since A_2 must be hit by a (-1)-curve, we deduce that $E_2 \cdot A_2 = 1$. We now check that E_1 and E_3 are disjoint from A_1. If $E_1 \cdot A_1 = 1$, then after blowing down E_1 the images of G and A_1 will intersect in a point and the image of C_i will pass through that point. When the image of C_i is eventually blown down, the intersection number of the images of G and A_2 will be 2, which is impossible for a pair of lines in $\mathbb{CP}^2$. If $E_3 \cdot A_1 = 1$, then the chain must have length one (to prevent the intersection number of the images of A_1 and D going above 3). But the then usual simple calculation shows that c must be 1 contradicting $c < 0$. We have thus checked that E_1 and E_3 are disjoint from A_1. It follows that, in order for the self-intersection number of the image of A_1 to increase to 1, we must have that E_2 intersects the string in the penultimate curve of the chain to get blown down. Suppose that $E_2 \cdot C_j = 1$. Now if $l < k$, then it is easy to see that we must have $k = 2$, $l = 1$ and $j = 2$. But then, after completing the blowing down process, the intersection number of the images of A_1 and D will be 2, a contradiction. Thus we must have $l = k$. It follows that we must have $j = 1$ or 2. If $j = 1$, then we must have $i = 2$, and if $j = 2$, then we must have $i = 1$. The blowing down process now fixes $c, c_1, \ldots, c_k$, which depends only on k and is independent of j, giving $c = -k+2, c_1 = -3, c_2 = -4$ and $c_3 = \ldots = c_k = -2$. The two possible configurations of the curves E_1, E_2, E_3 (providing the same dual graphs) are shown by Figure 21(c) and (d), both leading to the same configuration (e) after all the blow-downs are performed. □

Lemma 3.24 *The configuration of curves given by the graph Γ' of Figures 20(b) or 21(b) does exist in $\mathbb{CP}^2 \# (|\Gamma'| - 1)\overline{\mathbb{CP}^2}$. Consequently the singu-*

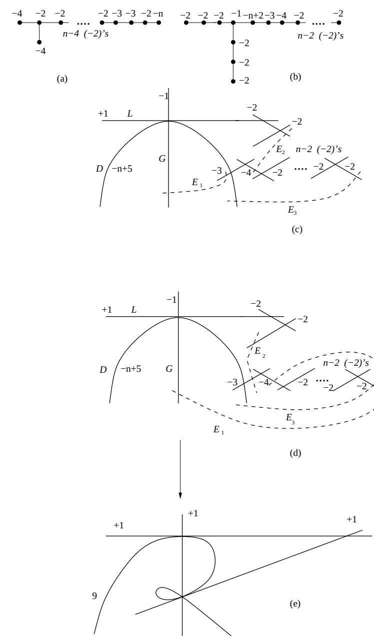

Figure 21: The further one-parameter family in $\mathcal{B}_2$ with rational homology disk filling.

larities with resolution graphs given by Figure 20(a) *or 21*(a) *do admit rational homology disk smoothings.*

Proof Consider first the outcome of Case II of the proof of Proposition 3.23. The nodal cubic curve with the tangent at one of its inflection points has been given in the proof of Lemma 3.4 already. Adding to it the line $\{x + z = 0\}$, which intersects the cubic curve once transversally (at the point $[0 : 1 : 0]$) and once tangentially (at $[-1 : 0 : 1]$), and the line $\{y = 0\}$ joining this tangency $[-1 : 0 : 1]$ with the node $[0 : 0 : 1]$, we get a configuration, from which appropriate repeated blow-ups provides an embedding of the dual configuration in an appropriate rational surface.

For Case III of the proof of Proposition 3.23, consider the cubic and its tangent as it is given in Lemma 3.4, together with the line $\{x + z = 0\}$ as above, and add $\{x = y\}$ (which is tangent to one of the branches of the cubic at its double point). The appropriate sequence of blow-ups provides a curve configuration in $\mathbb{CP}^2\#(|\Gamma'| - 1)\overline{\mathbb{CP}^2}$ with the desired intersection pattern Γ'.

Therefore Pinkham's Theorem 2.12 applies in both cases and shows the existence of the required smoothings. $\square$

Proof of Theorem 3.22 As usual, the implication (1) $\Rightarrow$ (2) follows from general principles, while Proposition 3.23 provides (2) $\Rightarrow$ (3). Finally, Lemma 3.24 implies (3) $\Rightarrow$ (1). $\square$

Remark 3.25 The graphs found in Case II are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf. The graphs of Case III are given by blowing up the edge connecting the (-1)-vertex to the leaf, except in the last blow-up. Notice that the graphs found in Case III already appeared in the family $\mathcal{M}$: compare with Figure 1(g) with the choice $p = 2, r = 1$ and $q = n - 4$.

After this long preparation we are ready to the give the proof of one of the main results of the paper.

Proof of Theorem 1.4 Consider a small Seifert singularity S_Γ. Since a smoothing of S_Γ provides a weak symplectic filling of the Milnor fillable contact structure (Y_Γ, ξ_Γ) of the link, the implication (1) $\Rightarrow$ (2) follows.

Now suppose that (2) holds for S_Γ. According to Theorem 2.9 then $\Gamma \in \mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Since $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \subset \mathcal{QHD}_3$ by definition (as the graphs of Figure 1), we only need to consider graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. The combination of (2) $\Rightarrow$ (3) of Theorems 3.1, 3.6, 3.10, 3.14, 3.18 and 3.22 verifies the implication (2) $\Rightarrow$ (3) of Theorem 1.4.

Finally, if $\Gamma \in \mathcal{QHD}_3$ is in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$ then [17] (cf. also Section 5) shows that the corresponding singularity admits a rational homology disk smoothing. If $\Gamma \in \mathcal{QHD}_3$ is given by one of the diagrams of Figure 2 then one of the implications (3) $\Rightarrow$ (1) of Theorems 3.1, 3.6, 3.10, 3.14, 3.18 or 3.22 verifies the implication (3) $\Rightarrow$ (1) of Theorem 1.4.

Notice that by a result of Laufer [6] all the graphs in $\mathcal{QHD}_3$ are taut: according to [6] a three-legged graph is taut if (a) the framing of the central vertex is ≤ -3 or (b) the framing of the central vertex is -2 and at least two arms of the graph are of length one. With this last observation the proof is complete. $\square$

4 Seifert singularities

Next we turn to an examination of generic Seifert singularities. According to the main result of [17], however, if a Seifert singularity admits a rational homology disk smoothing (or the Milnor fillable contact structure on its link admits a rational homology disk filling) then the valency of the central vertex is at most four. The three-legged case was analyzed in the previous section, so now we will focus on the case of four-legged graphs. Once again, it follows from [17] that we only need to consider graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

The family $\mathcal{C}$

We start by considering the four-legged graphs in the family $\mathcal{C}$. The generic

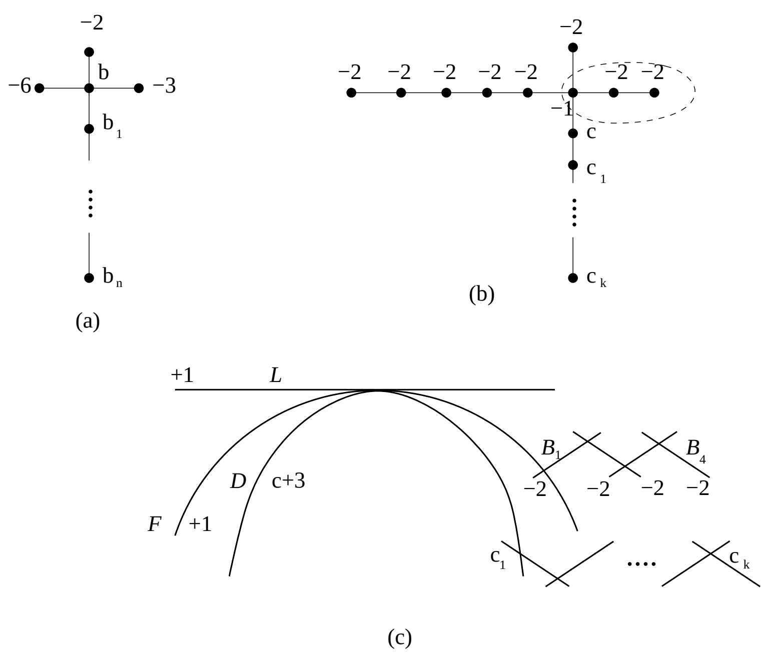

Figure 22: The four-legged graphs in $\mathcal{C}$.

four-legged member Γ of $\mathcal{C}$ is given in the first diagram of Figure 22, with the dual graph given by Figure 22(b).

Theorem 4.1 *Suppose that Γ is a plumbing graph as in Figure 22(a), and assume that $b \leq -3$. Then the following three statements are equivalent:*

(1) *There is a singularity S_Γ with resolution graph Γ which admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

(3) *for the graph Γ given by Figure 22(a) $b = -3$, $b_1 = \ldots = b_{n-1} = -2$ and $b_n = -n$ for some integer $n \geq 2$.*

Again, before starting the proof we list a few useful observations. After three blow-downs we obtain the configuration K depicted in the third picture in Figure 22. The horizontal $(+1)$-curve will be denoted L and the two curves which are triply tangent to L will be denoted F and D, with D being the innermost curve. Also the chain of (-2)-curves connected to the curve F will be denoted $B_1, \ldots, B_4$, with B_1 intersecting F and the chain of curves intersecting D will be denoted $C_1, \ldots, C_k$, with C_1 intersecting D. Suppose that X is a weak symplectic rational homology disk filling of (Y_Γ, ξ_Γ). As before, let $W_{\Gamma'}$ denote a regular neighbourhood of K for the dual graph Γ' and let Z denote the result of gluing X with $W_{\Gamma'}$. Then Z will be a closed symplectic 4-manifold containing the configuration K. We will blow down (-1)-curves disjoint from L to obtain $(\mathbb{CP}^2, L)$ with the images of F and D being cubics.

Let $\mathcal{J}_K$ denote the nonempty set of tame almost complex structure on Z with respect to which all the curves of K are pseudoholomorphic. Choose an almost complex structure J which is generic in $\mathcal{J}_K$. If we blow down all J-holomorphic (-1)-curves away from L, we can show that the strings $B_1, \ldots, B_4$ and $C_1, \ldots, C_k$ are transformed into configurations of curves which can be sequentially blown down. An elementary homological computation shows that (since X is a rational homology disk) there must be precisely two (-1)-curves, say E_1 and E_2, in the complement of L that are not contained in the strings $B_1, \ldots, B_4$ and $C_1, \ldots, C_k$. Since the string $B_1, \ldots, B_4$ must be transformed into a configuration which can be sequentially blown down after blowing down E_1 and E_2, it follows that at least one of these (-1)-curves must intersect $B_1 \cup \cdots \cup B_4$. Assume, without loss of generality, that E_1 intersects $B_1 \cup \cdots \cup B_4$.

Proposition 4.2 *By assuming the existence of the rational homology disk filling X we get that E_1 intersects D, F and B_4, while E_2 intersects D and C_k. The framings then are given by $c = -k-3$ and $c_1 = \ldots = c_k = -2$.*

Proof If $E_1 \cdot B_2 = 1$ or $E_1 \cdot B_3 = 1$, then blowing down E_1 and then sequentially blowing down the images of B_2 and B_3 leads to a $(+1)$-curve (the image

of B_1 or B_4) in the complement of L, contradicting Lemma 2.6. Hence we can assume that either $E_1 \cdot B_1 = 1$ or $E_1 \cdot B_4 = 1$.

Case I: Suppose that $E_1 \cdot B_1 = 1$. Note first that $E_1 \cdot F = 0$. Indeed, suppose that $E_1 \cdot F \geq 1$. If $E_1 \cdot F > 1$, then blowing down E_1 would lead to a point on the image F' of F under the blowing down map through which at least two branches of F' pass. Also the intersection number of the image B_1' of B_1 and F' will be at least three. By perturbing the almost complex structure slightly, we can assume that B_1' and F' intersect transversely. Then blowing down B_1' we see that the image F'' of F' will have two singularities, which by Lemma 2.7 contradicts the fact that F'' will eventually blow down to a cubic in $\mathbb{CP}^2$. A similar contradiction arises if $E_1 \cdot F = 1$, after blowing down both E_1 and B_1'. There are now two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Note that $E_1 \cdot (C_1 \cup \cdots \cup C_k) > 1$ is impossible by Corollary 2.5.

IA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Suppose that $E_1 \cdot C_i = 1$. After blowing down E_1 and then sequentially blowing down the images of $B_1, \ldots, B_4$, observe that the image C_i' of C_i will be 4-fold tangent to the image F' of F. Perturbing the almost complex structure, we may assume that C_i' intersects F' transversely. Eventually C_i' will get blown down and this will create a singularity on the image of F that is not allowed for a cubic in $\mathbb{CP}^2$, since the link of its singularity has four components, providing the desired contradiction.

IB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

We have $E_1 \cdot D = 0$ or $E_1 \cdot D = 1$. $E_1 \cdot D > 1$ is not allowed as blowing down E_1, then perturbing the almost complex structure so that B_1', the image of B_1, and D', the image of D, intersect transversely and then blowing down B_1' would create two nodes on the image of D', contradicting Lemma 2.7. After blowing down E_1 and then sequentially blowing down the images of $B_1, \ldots, B_4$, the intersection number of the images F' and D' of F and D, respectively, will be either 3 or 7. Now, by arguing as in the proof of Proposition 3.3, we can show that E_2 must intersect the last curve C_k in the string $C_1, \ldots, C_k$ and the curve D'. E_2 must also intersect F', otherwise, after the blowing down process has been carried out, the image of F' would be nonsingular and rational, which is impossible for a cubic in $\mathbb{CP}^2$. In fact, it is necessary that $E_2 \cdot F' = 2$, otherwise the image of F' will either be smooth or have the wrong type of singularity. Also it is necessary that the string $C_1, \ldots, C_k$ be empty, otherwise, after blowing down E_2, when the image of C_k is collapsed a further singularity will be introduced in the image of F'. Now the condition that D'

gets blown to a rational cubic in $\mathbb{CP}^2$ forces us to have $E_2 \cdot D' = 2$. Blowing down E_2, we see now that the intersection number of the images of D' and F' will be either 7 or 11 (depending on $E_1 \cdot D = 0$ or 1), which is impossible for a pair of irreducible cubic curves in $\mathbb{CP}^2$. In conclusion, we found that $E_1 \cdot B_1 = 1$ leads to contradiction, hence we can consider

Case II: $E_1 \cdot B_4 = 1$. As before, we distinguish two cases according to the intersection of E_1 with the chain $C_1 \cup \ldots \cup C_k$.

IIA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Suppose that $E_1 \cdot C_i = 1$. Note that $E_1 \cdot F = 0$, otherwise the image of F after completing the blowing down process would have more than one singular points. For a similar reason, $E_1 \cdot D$ must also be 0. We now divide $E_1 \cdot C_i = 1$ into three cases.

(i) Suppose that $i = 1$, i.e., E_1 intersects the chain in the curve intersecting D. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$ and then the images of $C_1, \ldots, C_l$ until the resulting string $C'_{l+1}, \ldots, C'_k$ attached to D', the image of D, is minimal, that is, contains no (-1)-curves. Let F' denote the image of F. Then $F' \cdot D' = l + 2$, where $0 \le l \le k$. First suppose that $l < k$. Then, by arguing as in the proof of Proposition 3.3, one can show that E_2 must intersect the last curve C'_k of the string $C'_{l+1}, \ldots, C'_k$ and the curves F' and D', each once transversally. Now blow down E_2 and then sequentially blow down the images of $C'_k, \ldots, C'_{l+1}$. Then the images of F' and D' will be nodal curves and for the intersection number of them to be 9 we require that $k = 2$. However, to make the self-intersection number of the image of F' equal 9 we require that $k = 3$. This contradiction show that the case $l < k$ cannot occur. Now suppose that $l = k$. Then to introduce singularities of the right type into the images of the curves F' and D' we require that $E_2 \cdot F' = 2$ and $E_2 \cdot D' = 2$. A simple check now shows that, as before, to make the intersection number of the images of F' and D' 9 we require $k = 2$ and to make the image of D' have self-intersection number 9 we require $k = 3$, again a contradiction.

(ii) Suppose next that $1 < i < k$ $(k \ge 3)$. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$. Suppose first that the image C'_i of C_i under the blowing down map is not a (-1)-curve. Then, arguing as in the proof of Proposition 3.3, one can show that E_2 must intersect the last curve C_k in the string attached to D and it must necessarily intersect F', the image of F. It follows that $i = 1$, otherwise, after blowing down E_2 and then sequentially blowing down the images of $C_k, \ldots, C_1$, the image of F' would have more than

one singularity, contradicting Lemma 2.7. Since $i > 1$ is assumed, we reached a contradiction. Thus C_i' must be a (-1)-curve. Now blow down C_i'. Note that the images of the curves C_{i-1} and C_{i+1} must be the last two curves (in some blowing down process) of the string attached to D to get blown down, otherwise the image of F' after completing the blowing down process will have more than one singular point, a contradiction. Now there are two cases to consider: $E_2 \cdot F' = 0$ or $E_2 \cdot F' = 1$.

Suppose that $E_2 \cdot F' = 0$. Then it is easy to see that after the blowing down process has been carried out, the image of F' will have self-intersection number 8, which contradicts the fact that F should blow down to a cubic in $\mathbb{CP}^2$.

Suppose that $E_2 \cdot F' = 1$. Then E_2 must be disjoint from the string attached to D. In order to make D singular, $E_2 \cdot D$ must necessarily be 2. It is now easy to check that, after carrying out the blowing down process, the intersection number of the images of the curves F and D will be less than 9, which contradicts the fact that they should blow down to a pair of cubics in $\mathbb{CP}^2$.

(iii) Finally assume that $i = k$ $(k \geq 2)$. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$ and then the images of $C_k, \ldots, C_{l+1}$ until the resulting string $C_1', \ldots, C_l'$ attached to D' (the image of D) is minimal. If a nonempty string remains, then, as before, E_2 must intersect the last curve C_l' in the string and the curves F', the image of F, and D', each once transversally. Then blowing down E_2 and then the image of C_l', we find that l must be 1, otherwise the image of F', after completing the blowing down process, would have more than one singular point, contradicting Lemma 2.7. It follows that the intersection number of the images of F' and D', after completing the blowing down process, will be 8, contradicting the fact that they should blow down to a pair of cubics in $\mathbb{CP}^2$.

If $l = 1$, that is the whole string attached to D gets sequentially blown down after blowing down E_1, then one can check that the intersection numbers of E_2 and the images of F' and D' must both be 2. Again it follows that, after completing the blowing down process, the intersection numbers of the images of F' and D' will be 8, a contradiction. This completes **IB** and hence we conclude that

IIB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

We claim that $E_1 \cdot F = 1$. To see this, suppose, for a contradiction, that $E_1 \cdot F = 0$. Then we have $E_1 \cdot D = 0$ or 1. Blow down E_1 and then sequentially blow down the images of the curves $B_4, \ldots, B_1$. Then the image F' of F will still be smooth. It is thus necessary to have $E_2 \cdot F' = 2$, otherwise the image

of F will be smooth or have the wrong type of singularity. But then the string $C_1, \cdots, C_k$ must be empty, otherwise E_2 would have to intersect it and thus blowing down would create additional singular points on the image of F, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be less than 9, a contradiction. This verifies $E_1 \cdot F = 1$.

Now blowing down E_1 and then sequentially blowing down the B_i, we find that the image of F becomes a rational curve with a single nodal point and having self-intersection number 9. It follows that E_2 cannot intersect F and that E_1 must intersect D once transversally. Let F', D' denote the images of F and D, respectively, after blowing down E_1 and the B_i. It is then easy to check that $F' \cdot D' = 9$. Now the only possibility for E_2, by the argument in the proof of Proposition 3.3, is that $E_2 \cdot C_k = 1$ and $E_2 \cdot D = 1$. For each value of k, the blowing down process now fixes c and $c_1, \ldots, c_k$, which (with $n = k+2$) must be as in Figure 23(a). □

Lemma 4.3 *There does exist a configuration of curves in $\mathbb{CP}^2$ having the intersection pattern given in Figure 23(d). Consequently, there are singularities with resolution graphs given in Figure 23(a) which admit rational homology disk smoothings.*

Proof Let L be the line $\{z=0\}$ in $\mathbb{CP}^2$ and let R_1 and R_2 be the cubics given by the equations $f_1(x,y,z) = y^2z - x^3 - x^2z$ and $f_2(x,y,z) = y^2z + \frac{1}{2}xyz + yz^2 - \frac{9}{8}x^3 - 2x^2z - xz^2$, respectively. The curves R_1 and R_2 are rational nodal cubics with nodes at $[0:0:1]$ and $[-\frac{2}{3} : -\frac{1}{3} : 1]$, respectively. It is easy to check that both R_1 and R_2 are triply tangent to L at the point $[0:1:0]$ and are also triply tangent to each other at $[0:1:0]$ and have intersection multiplicity 6 at the point $[0:0:1]$. Therefore the existence of the configuration of curves depicted by Figure 23(d) is verified, from which the appropriate sequence of blow-ups shows the existence of the embedding of curves with intersections given by the graph Γ' of Figure 23(b) in $\mathbb{CP}^2 \# (|\Gamma'|-1)\overline{\mathbb{CP}^2}$. The existence of the smoothing of a (weighted homogeneous) singularity with resolution graph of Figure 23(a) then follows from Pinkham's Theorem 2.12. □

Proof of Theorem 4.1 As before, the implication $(1) \Rightarrow (2)$ follows from general principles, $(2) \Rightarrow (3)$ is a direct consequence of Proposition 4.2 and $(3) \Rightarrow (1)$ is implied by Lemma 4.3. □

Remark 4.4 The graphs found in this case are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf.

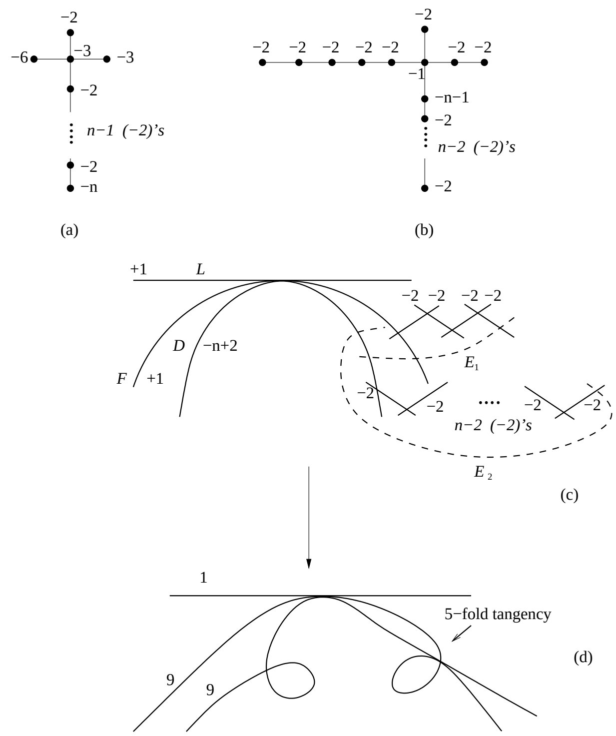

Figure 23: The one-parameter family of 4-legged graphs in $\mathcal{C}$ with rational homology disk filling.

Notice also that the existence of the smoothings was already proved in [17, Example 8.7].

The family $\mathcal{B}$

We next consider four-legged graphs in the family $\mathcal{B}$: the generic four-legged member of this family is given by Figure 24(a) with the dual graph given by Figure 24(b).

Theorem 4.5 *Suppose that* Γ *is a plumbing graph as in Figure 24(a) and assume that* $b \leq -3$. *Then the following three statements are equivalent:*

(1) *There is a singularity* S_Γ *with resolution graph* Γ *which admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold* (Y_Γ, ξ_Γ) *admits a weak symplectic*

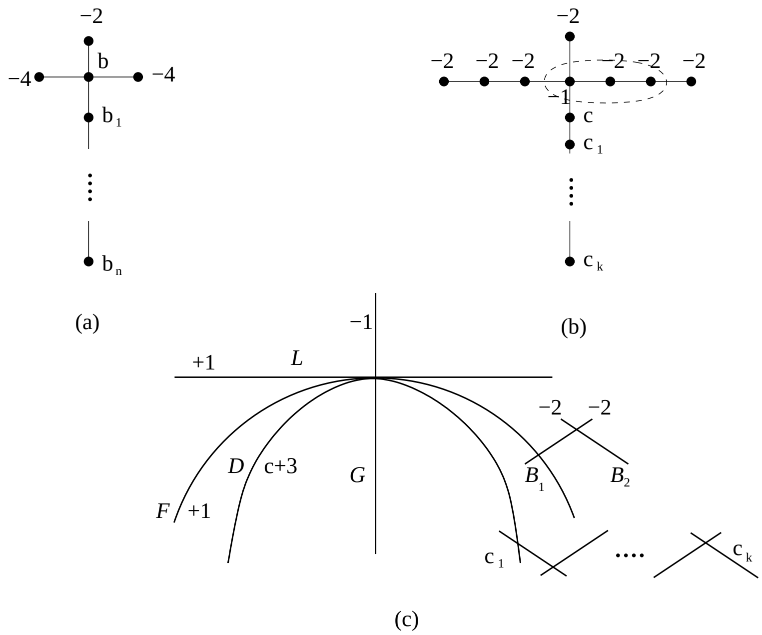

Figure 24: The 4-legged graphs in $\mathcal{B}$.

rational homology disk filling, and

(3) *for the graph* Γ *given by Figure 24(a)* $b=-3$, $b_1=\ldots=b_{n-2}=-2$, $b_{n-1}=-3$ *and* $b_n=-n$ *for some integer* $n\geq 2$.

After three blow-downs we obtain the configuration K depicted in the third picture in Figure 24. Suppose that Z is the closed symplectic 4-manifold we get by gluing the compactifying divisor $W_{\Gamma'}$ (containing K) to a weak symplectic rational homology disk filling of (Y_Γ,ξ_Γ). Then it is easy to check that there must be three (-1)-curves, say E_1,E_2,E_3, not contained in the strings B_1,B_2 and $C_1,\ldots,C_k$, such that, after blowing down these three (-1)-curves, the images of the curves in the strings B_1,B_2 and $C_1,\ldots,C_k$ can be sequentially blown down and in the process F and D will be transformed to a pair of cubics in $\mathbb{CP}^2$ and the images of G and L will be lines.

Since in the blowing down process the string B_1,B_2 will eventually transformed into a string which can be sequentially blown down, one of the (-1)-curves E_1,E_2,E_3, must intersect $B_1\cup B_2$. Renumbering the curves if necessary, we may assume that this curve is E_1.

Proposition 4.6 *Under the hypothesis of the existence of a rational homology disk filling, we get that E_1 intersects D, F and B_2, E_2 intersects F, G and C_2, while E_3 intersects D and C_k. The corresponding framings are given as $c = -k-2$, $c_1 = -3$ and $c_2 = \ldots = c_k = -2$.*

Proof Note that E_1 must be disjoint from G, otherwise blowing down E_1 and then sequentially blowing down the images of B_1 and B_2 the image of G would be either singular or would have self-intersection number 2, which contradicts the fact that G should blow down to a line in $\mathbb{CP}^2$. Since one of the E_i must necessarily intersect G we may assume that $E_2 \cdot G = 1$. We now consider the two possibilities: $E_1 \cdot B_i = 1$ for $i = 1, 2$.

Case I: $E_1 \cdot B_1 = 1$.

The curve E_1 must necessarily be disjoint from F, otherwise the image of F after completing the blowing down process would have more than one singular point which is impossible for a cubic in $\mathbb{CP}^2$. We consider the two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Suppose that $E_1 \cdot C_i = 1$. Note that the image of C_i must be the last curve of the string attached to D to get blown down, since blowing down the the image of C_i will make the image of F singular so that if there are any remaining curves in the string then these will create additional singularities on the image of F when they are blown down, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Then the condition that G blows down to a $(+1)$-curve in $\mathbb{CP}^2$, forces us to have $E_3 \cdot G = 1$. But then necessarily $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Thus the string $C_1, \ldots, C_k$ must have length 1. Now E_2 and E_3 must necessarily intersect F, each once transversally, otherwise the intersection number of the images of F and G will not be 3. It is also necessary that the intersection number of one of E_2 or E_3 and D be 2 and the other be 0 to meet the requirements that the image of D be singular and that the images of D and G have intersection number 3. But then after completing the blowing down process we will find that the images of D and F have intersection number 7, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Note that E_2 must necessarily intersect C_i, the last curve in the string to get blown down, otherwise the image of G after repeatedly blowing down will have self-intersection number greater than 1, a contradiction. Note also that E_2 must be disjoint from F, otherwise

blowing down the image of C_i will lead to a triple point on the image of F, a contradiction. Now consider the (-1)-curve E_3. If E_3 intersects $C_1 \cup \cdots \cup C_k$, then E_3 will be disjoint from F. In such a case, after completing the blowing down process, the image of F will be a 7-curve, a contradiction. If E_3 is disjoint from $C_1 \cup \cdots \cup C_k$, then $E_3 \cdot F$ can be 0 or 1. In either case, after completing the blowing down process, the image of F will have self-intersection number at most 8, again a contradiction. This argument concludes the analysis of the case $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

IB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$ as well. As before, it implies that $E_3 \cdot G = 1$. It follows that E_1, E_2, E_3 will be disjoint from $C_1 \cup \cdots \cup C_k$. But this means that the string must be empty, which is never the case.

Suppose now that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Then E_2 must intersect the last curve of the string to get blown down. Also we must necessarily have $E_3 \cdot G = 0$. If E_3 is disjoint from $C_1 \cup \cdots \cup C_k$, then the string must have length 1. It follows that, after completing the blowing down process, the intersection number of the images of D and G will be either 2 or 4, depending on whether $E_2 \cdot D = 0$ or 1, a contradiction in both cases. So we may assume that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Note that the only way an appropriate singularity on the image of D can arise is if $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$ and $E_2 \cdot C_1 = 1$. Note also that we necessarily have $E_2 \cdot F = 1$, otherwise the intersection number of the images of F and G will not be 3. If $E_3 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, a contradiction. If $E_3 \cdot F = 1$, then after completing the blowing down process, the intersection number of the images of F and G will be 4, again a contradiction. This last observation concludes the discussion of Case I and shows that $E_1 \cdot B_1 = 1$ is not possible.

Case II: $E_1 \cdot B_2 = 1$.

Again we consider the two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IIA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Note that $E_1 \cdot F = 0$, otherwise when the image of C_i is eventually blown down the image of F will develop more than one singularity, a contradiction. For a similar reason we also have $E_1 \cdot D = 0$. Suppose that $E_1 \cdot C_i = 1$. We consider the possibilities for i.

(i) $i = 1$. Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Then the condition that the image of G, after completing the blowing down process, be a $(+1)$-curve forces us to have $E_3 \cdot G = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Also, the condition that the images of F and D have nodes and that the intersection numbers of the images of F and G, and D and G be 3 forces us to have $E_2 \cdot F = 2$, $E_2 \cdot D = 0$ and $E_3 \cdot F = 0$, $E_3 \cdot D = 2$, or vice-versa. Finally, the condition that F will have self-intersection number 9 forces us to have $k = 3$. But then it follows that the intersection number of the images of F and D, after completing the blowing down process, will be 6, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Then E_2 will intersect the last curve of the string to get blown down. Note that $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and G will be greater than 3, a contradiction. Similarly $E_2 \cdot F = 0$. Note also that E_3 is necessarily disjoint from G. Thus if E_3 is also disjoint from the string or from D, it follows that the intersection number of D and G after completing the blowing down process will be 2, a contradiction. Thus E_3 necessarily intersects the string and D. In fact, we require that $E_3 \cdot C_k = 1$. Now the condition that the image of F have a singularity forces us to have $E_3 \cdot F = 1$. Also, the condition that the image of F have self-intersection number 9 forces us to have $k = 3$. However, if $k = 3$, then the intersection number of the images of F and D, after completing the blowing down process, will be 10, a contradiction.

(ii) $1 < i < k$ $(k \geq 3)$. If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then, as before, we require that $E_3 \cdot G = 1$, $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. It follows that we must have $k = 3$, otherwise, after completing the blowing down process, the image of F will either have a singularity of multiplicity greater than two or will have more than one singular point, neither of which is permitted for a cubic in $\mathbb{CP}^2$. Now the condition that the images of F and G have intersection number 9 forces us to have $E_2 \cdot F = E_3 \cdot F = 1$. But then the image of F, after completing the blowing down process, will have self-intersection number 10, a contradiction. Thus $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve of the string that gets blown down. Note that, as in the previous case, $E_2 \cdot F = 0$, $E_2 \cdot D = 0$.

Suppose that $C_i \cdot C_i = -4$. Then the image of C_i will be a (-1)-curve, after blowing down E_1 and then sequentially blowing down the images of B_2, B_1. It follows that the images of C_{i-1}, C_{i+1} must be the last two curves of the string attached to D to get blown down. Since $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$, note that, as before, we require $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$. Note that $E_3 \cdot F = 0$, otherwise the image of F after completing the blowing down process would have more than one singular points,

a contradiction. Now, after completing the blowing down process, we find that the intersection number of the images of D and F will be 8, a contradiction.

Suppose that $C_i \cdot C_i < -4$. Then after blowing down E_1 and then sequentially blowing down B_2, B_1, the image of C_i will not be a (-1)-curve. As before, we can show that $E_3 \cdot C_k = 1$, $E_3 \cdot D = 1$. The condition that F become singular forces us to have $E_3 \cdot F = 1$. Now after completing the blowing down process we see that the F will have more than one singularity, since $i > 1$, a contradiction.

(iii) $i = k$ $(k \geq 2)$. If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then, as before, we require that $E_3 \cdot G = 1$, $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. To obtain the correct types of singularities on the images of F and D and to meet the requirement that the intersection numbers of the images of F and G, and D and G, after completing the blowing down process, be 3, we require that $E_2 \cdot F = 2$, $E_3 \cdot F = 0$ or $E_2 \cdot F = 0$, $E_3 \cdot F = 2$ and likewise for D. It follows that after completing the blowing down process the intersection number of the images of F and D will be 8, a contradiction. So $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve of the string that gets blown down.

Suppose that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$. Then since $E_3 \cdot G = 0$, after completing the blowing down process the intersection number of the images of D and G will be 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ and $E_3 \cdot D = 1$. Similarly we can check that $E_3 \cdot F = 1$.

Suppose that $E_3 \cdot C_j = 1$ for $j < k$. Blow down E_1, E_2, E_3 and then sequentially blown down the images of B_2, B_1. Note then that, after the images of $C_k, C_{k-1}, \ldots, C_j$ have been sequentially blown down, the image of F will become singular. Also after the images of $C_j, C_{j-1}, \ldots, C_2$ have been sequentially blown down the image of D will become singular. Since the images of F and D should have exactly one singularity, the image of C_j must necessarily be the last curve of the string to get blown down. It follows that j must be 2. It is now easy to check that, after the blowing down process has been completed, the intersection number of the images of F and D will be 8, a contradiction.

Suppose that $E_3 \cdot C_k = 1$. Then once the image of C_k is blown down the image of F will become singular. It follows that k must be 1, contrary to assumption.

IIB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we must have $E_3 \cdot G = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. It follows that the string $C_1, \ldots, C_k$ must be empty, which is never the case. So $E_2 \cdot (C_2 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve that gets blown

down. We thus necessarily have $E_3 \cdot G = 0$.

Suppose that $E_1 \cdot F = 0$. If $E_2 \cdot F = 0$ also, then the only way that the image of F can have the correct type of singularity is if $E_3 \cdot F = 2$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. But then, after completing the blowing down process, we find that the intersection number of the images of F and G will be 2, a contradiction. So $E_2 \cdot F = 1$. There are now two ways that the image of F can have the correct type of singularity: if $E_3 \cdot F = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or if $E_3 \cdot F = 2$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. In the former case, after completing the blowing down process, the intersection number of the images of F and G will be 4, a contradiction. In the latter case, after completing the blowing down process, the intersection number of the images of D and G will be either 2 or 4 depending on whether $E_2 \cdot D = 0$ or 2, a contradiction in either case.

Suppose that $E_1 \cdot F = 1$. If $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of F and G will be either 2 (if $E_3 \cdot F = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$) or 1 (if $E_3 \cdot F = 0$ or $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$), a contradiction in either case. So $E_2 \cdot F = 1$. Note now that if $E_3 \cdot F = 1$, then the self-intersection number of the image F, after completing the blowing down process, will be greater than 9, which is not possible for a cubic in $\mathbb{CP}^2$, implying that $E_3 \cdot F = 0$. Also if $E_2 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of D and G will be greater than 3, a contradiction, hence we conclude that $E_2 \cdot D = 0$. Next note that if $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$, then since $E_3 \cdot G = 0$, after completing the blowing down process, the intersection number of the images of D and G will be 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$ and $E_2 \cdot C_1 = 1$. Also if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of D and F will be 5, a contradiction. So we must have $E_1 \cdot D = 1$. Thus the three (-1)-curves E_1, E_2, E_3 must be as given by the Proposition (cf. also Figure 25(c)). Finally, for each value of k, the blowing down process fixes c and $c_1, \cdots, c_k$, which (with $n = k+1$) must be as given in Figure 25(a). □

Lemma 4.7 *There does exist a configuration of curves in $\mathbb{CP}^2$ shown by Figure 25(d), hence for the graph Γ' given by Figure 25(b) there are curves in $\mathbb{CP}^2 \#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$ intersecting each other according to Γ'. Consequently, there are singularities with resolution graphs given in Figure 25(a) which admit rational homology disk smoothings.*

Proof Let L and R_1 be as before and let M be the line $\{x + z = 0\}$ and R_3 be the cubic given by the equation $f_3(x, y, z) = y^2z + 2xyz + 2yz^2 - 2x^3 -$

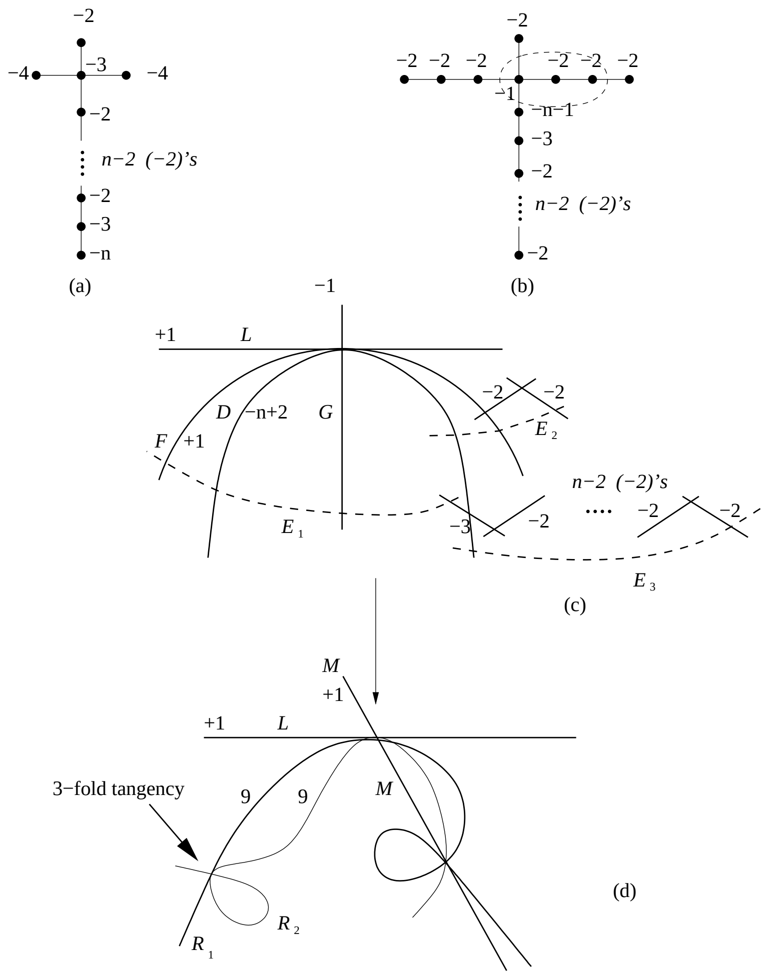

Figure 25: The one-parameter family of 4-legged graphs in $\mathcal{B}$ with rational homology disk filling.

$4x^2z - 2xz^2$. The curve R_3 is a rational nodal cubic with a node $[-1:0:1]$. One can check that L, R_1 and R_3 are pairwise triply tangent at $[0:1:0]$. Also R_1 and R_3 intersect at $[0:0:1]$ with intersection multiplicity 4 and at $[-1:0:1]$ with intersection multiplicity 2. Furthermore, M passes through the point $[0:1:0]$ and is tangent to R_1 at $[-1:0:1]$. Therefore the existence of the configuration of curves depicted by Figure 25(d) is verified, from which the appropriate sequence of blow-ups verifies the existence of the embedding of curves with intersections given by Figure 25(b). The existence of

the smoothing of a (weighted homogeneous) singularity with resolution graph given by Figure 25(a) then follows from Pinkham's Theorem 2.12. □

Proof of Theorem 4.5 As before, the implication (1) $\Rightarrow$ (2) follows from general principles, (2) $\Rightarrow$ (3) is a direct consequence of Proposition 4.6 and (3) $\Rightarrow$ (1) is implied by Lemma 4.7. □

Remark 4.8 The graphs found in this case are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf. The existence of the smoothings was already proved (by slightly different means) in [17, Example 8.14].

The family $\mathcal{A}$

Finally we consider four-legged graphs in the family $\mathcal{A}$. The generic four-legged

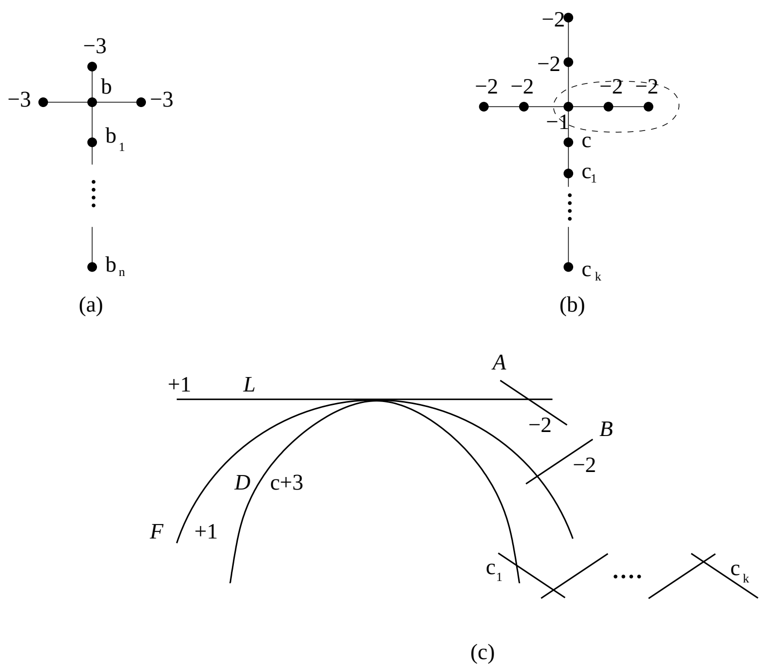

Figure 26: The 4-legged graphs in $\mathcal{A}$.

member Γ of $\mathcal{A}$ is given in the first picture in Figure 26 with the dual graph in the second.

Theorem 4.9 *Suppose that Γ is a plumbing graph as in Figure 26(a) and assume that $b \leq -3$. Then the following three statements are equivalent:*

(1) *There is a singularity S_Γ with resolution graph Γ which admits a rational homology disk smoothing,*

(2) *the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ) admits a weak symplectic rational homology disk filling, and*

(3) *for the graph Γ given by Figure 24(a) $b = -3$, $b_1 = \ldots = b_{n-2} = -2$, $b_{n-1} = -4$ and $b_n = -n$ for some integer $n \geq 2$.*

After three blow-downs we obtain the configuration K indicated in the third picture in Figure 26. Suppose that Z is the closed symplectic 4-manifold we get by symplectically gluing the compactifying divisor $W_{\Gamma'}$ (containing K) to a weak symplectic rational homology disc filling of Y_Γ. Then it is easy to check that there must be three (-1)-curves, say E_1, E_2, E_3, not contained in the string $C_1, \ldots, C_k$, such that, after blowing down these three (-1)-curves, the image of B can be blown down and the images of the curves in the string $C_1, \ldots, C_k$ can be sequentially blown down so that in the process F and D are transformed to a pair of cubics in $\mathbb{CP}^2$ and the images of L and A are lines.

Since in the blowing down process B will be eventually transformed into a curve which can be blown down, one of the three (-1)-curves, call it E_1, must intersect B.

Proposition 4.10 *If a rational homology disk filling exists in the situation described above, then either Γ' blows down to a 3-legged graph (and was treated earlier), or E_1 intersects D, F and B, E_2 intersects A, D and F and E_3 intersects D and C_k. The corresponding framings in the latter case are given as $c = -k$, $c_2 = -3$ and $c_1 = c_3 = \ldots = c_k = -2$.*

Proof Note that if $E_1 \cdot A = 1$, then $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$, otherwise after blowing down E_1 and then the image of B, the image of A will become singular when the image of C_i is eventually blown down, where $E_1 \cdot C_i = 1$, which contradicts the fact that the image of A in $\mathbb{CP}^2$ will be a line. Thus at least one (-1)-curve different from E_1 should intersect A. Let us call this (-1)-curve E_2. We now begin the case-by-case analysis.

Case I: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Suppose that $E_1 \cdot C_i = 1$. In this case, by the argument above, we will necessarily have $E_1 \cdot A = 0$. Note that if $E_1 \cdot F = 1$, then after E_1 and the image of B

are blown down, the image F' of F will be singular. However, the intersection number of the image C_i' of C_i and F' will be 2. Thus when the image of C_i' is eventually blown down the image of F' have a second singularity, which contradicts the fact that it must eventually blow down to a cubic in $\mathbb{CP}^2$. Thus $E_1 \cdot F = 0$. Also, we must have $E_1 \cdot D = 0$, otherwise, after repeatedly blowing down, the image of D will eventually have a triple point, which contradicts the fact that the image of D in $\mathbb{CP}^2$ should also be a cubic.

Note that if $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we must have have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, since, after completing the blowing down process, the image of A should be a smooth curve of self-intersection number 1. Renumbering E_2 and E_3, if necessary, we may assume that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

Suppose that $E_2 \cdot C_j = 1$. Notice that, in the blowing down process, the image of C_j must either be the last curve of the string attached to D to get blown down or it must be the penultimate curve to get blown down, since otherwise, after the blowing process is complete, the self-intersection number of the image of A will be greater than 1, a contradiction.

(i) $i = 1$.

(ia) Suppose first that the image C_j is the last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Now, since $E_1 \cdot D = 0$, there are two ways that an appropriate singularity can appear on image of D: either $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$.

Suppose that $E_2 \cdot D = 1$. Then $E_3 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A would be greater than 3, a contradiction. We now have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$, otherwise the image of F, after completing the blowing down process, would be smooth and rational, which is a contradiction. Now the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 2$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 7, a contradiction. If $E_2 \cdot F = 1$, then $E_3 \cdot F = 0$, otherwise the intersection number of the images of F and A, after completing the blowing down process, would be greater than 3, a contradiction. Now, again, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, again a contradiction.

Suppose that $E_3 \cdot D = 2$. Then $E_2 \cdot D = 0$. We now have $E_2 \cdot F = 0$ or 1. If

$E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$. Now, as before, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction. If $E_2 \cdot F = 1$, then we must have $E_3 \cdot F = 0$. Thus, again, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces $k = 3$. And, this time, after completing the blowing down process, the intersection number of the images of F and D will be 7, again a contradiction.

(ib) The image of C_j is then the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$. Also, we must have $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A would be greater than 3, a contradiction. Similarly, we must have $E_2 \cdot F = 0$.

Suppose that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$. Then, after completing the blowing down process, the intersection number of the images of D and A will be at most 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. If $E_3 \cdot C_l = 1$ for $l < k$, then we must have $l = k - 1$ and $j = k$, otherwise, after completing the blowing down process, the image of D will have more than one singular point, a contradiction. However, if $l = k - 1$ and $j = k$, then, after completing the blowing down process, the intersection number of the images of D and A will be 2, a contradiction. So we must have $E_3 \cdot C_k = 1$. Also we must have $E_3 \cdot F = 1$, otherwise the image of F, after completing the blowing down process will be smooth, a contradiction. Now, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction.

(ii) $1 < i < k$ $(k \geq 3)$.

(iia) The image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. If k is greater than 2, then, after completing the blowing down process, the image of F will either have a point of multiplicity greater than 2 or have more than one singular point, neither of which is possible for a cubic in $\mathbb{CP}^2$. Thus we must have $k = 3$ and thus $j = 1$ or 3. Also, we must have $E_2 \cdot F = 0$, otherwise, after completing the blowing down process, the image of F will have a triple point, a contradiction. Furthermore, we must have $E_3 \cdot F = 1$, otherwise, after completing the blowing

down process, the intersection number of the images of F and A will be less that 3, a contradiction. Now the only way a singularity of the appropriate type can appear on the image of D is if $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$.

Suppose first that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of A and D will be greater than 3, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, which contradicts the fact that images of F and D in $\mathbb{CP}^2$ are a pair of cubics.

Suppose now that $E_3 \cdot D = 2$. Then we must have $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A will be greater that 3, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, a contradiction.

(iib) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$ and $E_2 \cdot D = E_2 \cdot F = 0$. Also, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$.

Suppose that $E_3 \cdot C_l = 1$ for $l < k$. Then the image of C_l must be the last curve of the string attached to D to get blown down. Indeed, it is easy to see that after the image of C_l is blown down, the image of the portion $C_{l+1}, \ldots, C_k$ of the the string must be a point, otherwise, after completing the blowing down process, the image of D will have more than one singular point. Thus we must have $i > l$ or $j > l$. In the former case, after the portion $C_l, \ldots, C_k$ of the the string has been collapsed to a point, the image of F will be singular and thus the image of C_l must be the last curve of the string to get blown down. In the latter case, since the image of C_j is the penultimate curve of the string to get blown down, C_l must be the last curve of the string to get blown down. Now again using the assumption that the image of C_j is the penultimate curve of the string to get blown down, we must have either $j < l$ or $j > l$. Suppose that $j < l$. Then we must have $i > l$. Also, we must have $E_3 \cdot F = 0$, otherwise, after completing the blowing down process, the image of F will have a singularity of multiplicity greater than 2, a contradiction. Now, after completing the blowing down process, the intersection number of the images of A and F will be 2, a contradiction. Suppose that $j > l$. Then, after completing the blowing down process, the intersection number of the images of A and D will be 2, again a contradiction.

Suppose that $E_3 \cdot C_k = 1$. Then we must have $E_3 \cdot F = 0$ or 1. Suppose that

$E_3 \cdot F = 0$. Then, in the blowing down process, the images of the curves C_{i-1} and C_{i+1} must be the last two curves of the string attached to D to get blown down. It follows that we must have $i = 2$. It is now easy to check that, after completing the blowing down process, the image of F will have self-intersection number 8, a contradiction. Suppose that $E_3 \cdot F = 1$. Then the image of C_i must be the last curve of the string to get blown down. It follows that we must have $i = 2$ and $j = 1$. We now find that, after completing the blowing down process, the intersection number of the images of F and A will be 2, a contradiction.

(iii) $i = k$ $(k \geq 2)$.

(iiia) The image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Also we must have $j = 1$. Now the only way a singularity of the appropriate type can appear on the image of D is if $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$.

Suppose that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$. Now we have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then it is easy to see that, after completing the blowing down process, the intersection number of the images of F and D will be 5, a contradiction. If $E_2 \cdot F = 1$, then one can check that, after completing the blowing down process, the intersection number of the images of F and D will be 8, again a contradiction.

Suppose that $E_3 \cdot D = 2$. Then we must have $E_2 \cdot D = 0$. Again we have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be 8, a contradiction. If $E_2 \cdot F = 1$, then we must have $E_3 \cdot F = 0$. In this case, after completing the blowing down process, the intersection number of the images of F and D will be 5, again a contradiction.

(iiib) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$ and $E_2 \cdot D = E_2 \cdot F = 0$. Also, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. Furthermore, we must have $E_3 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F would be smooth, a contradiction. Now note that if $l \neq 1$, then we must have $l = 2$ and $j = 1$, otherwise, after completing the blowing down process, the image of F will have more than one singular point, a contradiction. If $l = 1$, then C_1 must be the last curve to get blown down, otherwise, after completing the blowing down process, the image of D will have more than one singular point, a contradiction. Thus we must have $j = 2$. It now follows that, after completing the blowing down process, the intersection number of the images of D and A

will be 2, a contradiction. If $l = 2$ and $j = 1$, then C_2 must be the last curve to get blown down and in this case it follows that, after completing the blowing down process, the intersection number of the images of F and A will be 2, again a contradiction.

Case II: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IIA. $E_1 \cdot A = 1$.

Since we are assuming that $E_2 \cdot A = 1$ also, we will necessarily have $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$ and $E_3 \cdot A = 0$. Also, since the string $C_1 \ldots, C_k$ is nonempty for every 4-legged graph Γ in $\mathcal{A}$, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Now if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of D and A will be at most 2, a contradiction. It follows that we must have $E_1 \cdot D = 1$ and thus we must also have $E_2 \cdot D = 1$.

Suppose that $E_1 \cdot F = 1$. Then we must have $E_2 \cdot F = 0$. If $E_3 \cdot F = 0$ also holds, then, after completing the blowing down process, the self-intersection number of the image of F will be 6, a contradiction. So we must have $E_3 \cdot F = 1$ and k must be 2. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction.

Suppose that $E_1 \cdot F = 0$. Then we must have $E_2 \cdot F = 2$. Again we require $E_3 \cdot F = 1$ and $k = 2$. It thus follows again that, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction as before.

IIB. $E_1 \cdot A = 0$.

We may now assume $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$, since if $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we would necessarily have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, and we would just renumber the (-1)-curves. Suppose that $E_2 \cdot C_j = 1$. It follows that, in the blowing down process, the image of C_j is either the last curve of the string to get blown down or the penultimate curve to get blown down.

(i) Suppose first that the image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Since we are assuming that $E_1 \cdot A = 0$, we must have that $k = 1$. Now if $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of A and F will be at most 2, a contradiction. So $E_2 \cdot F = 1$ and thus $E_3 \cdot F = 1$ also. It follows that we must have $E_1 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F would be smooth or have more than one singularity, a contradiction in both cases.

Suppose that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$. Note also that we must have $E_1 \cdot D = 1$, otherwise, after completing the blowing down process, the intersection number of the images of F and D will be different from 9, a contradiction. It follows that D must have self-intersection number 2 and C_1 must have self-intersection number -2. It is easy to see that in this case Γ is just the unique three-legged graph in the family $\mathcal{A}$ with four vertices and we already know that in this case the corresponding contact 3-manifold (Y_Γ, ξ_Γ) admits a rational homology disk filling.

Suppose that $E_2 \cdot D = 0$. Then we must have $E_3 \cdot D = 2$. Again we can check that we must have $E_1 \cdot D = 1$. As in the previous case, it follows that D must have self-intersection number 2 and C_1 must have self-intersection number -2, and this case has already been considered.

(ii) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$. Note that if $E_2 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of A and D will be 4, a contradiction. Thus $E_2 \cdot D = 0$. Also we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$, otherwise, after completing the blowing down process, the intersection number of the images of A and D will be at most 2, a contradiction. Suppose that $E_3 \cdot C_l = 1$. Now if $l < k$, then we must have $k = 2$, $l = 1$ and $j = 2$. But then, after completing the blowing down process, the intersection number of the images of A and D will be 2, a contradiction. So $l = k$. It follows that we must have $j = 1$ or 2. Now note that if $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of A and F will be at most 2, a contradiction. So we must have $E_2 \cdot F = 1$. It also follows that we must have $E_3 \cdot F = 0$, otherwise, after completing the blowing down process, the intersection number of the images of A and F will be greater than 3, a contradiction. We now must have $E_1 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F will be smooth, a contradiction. For each value of k and for $j = 1, 2$, the blowing down process now fixes $c, c_1, \ldots, c_k$, which (with $n = k$) must be as in Figure 27. □

Lemma 4.11 *There does exist a configuration of curves in $\mathbb{CP}^2$ having the intersection pattern given in Figure 27(d). This shows that there are curves embedded in $\mathbb{CP}^2 \# (|\Gamma'| - 1)\overline{\mathbb{CP}^2}$ intersecting each other according to the plumbing graph Γ' given by Figure 27(b). In turn, this fact implies that for each graph of Figure 27(a) there is a singularity with that resolution graph which admits a rational homology disk smoothing.*

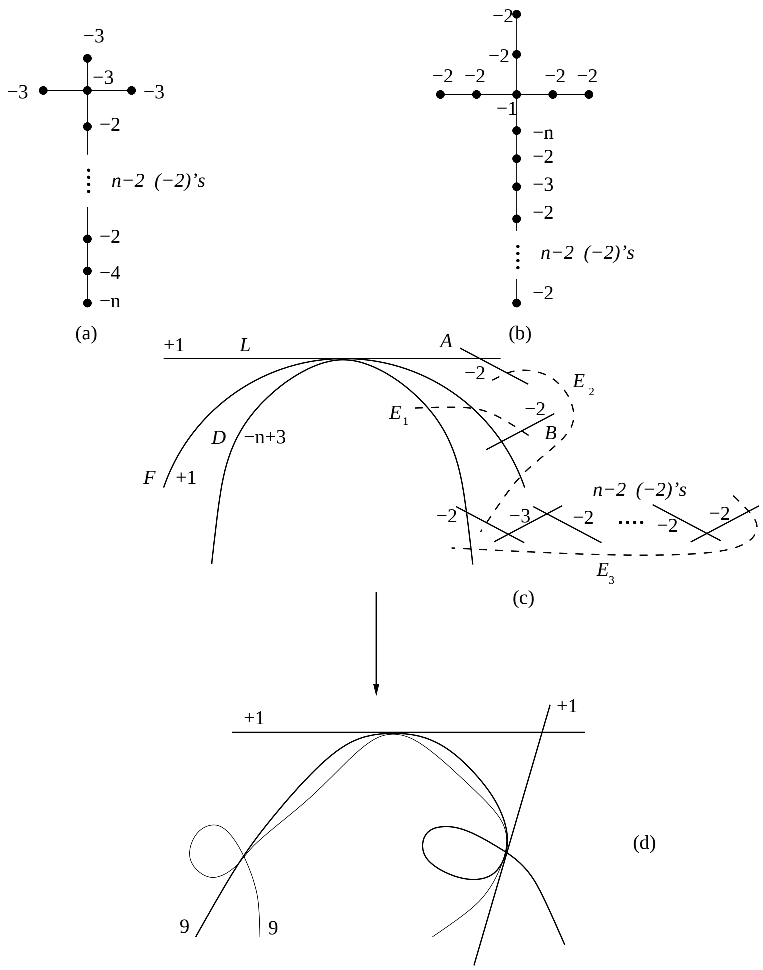

Figure 27: The one-parameter family of 4-legged graphs in $\mathcal{A}$ with rational homology disk filling.

Proof Let L and R_1 be as before; let N be the line $\{y-i\sqrt{3}(x+\frac{8}{9}z)=0\}$ and R_4 be the cubic given by the equation $f_4(x,y,z)=y^2z+(1-i\sqrt{3})xyz+\frac{4}{9}(3-i\sqrt{3})yz^2+\frac{1}{2}(-1+i\sqrt{3})x^3+(-2+i\sqrt{3})x^2z-\frac{4}{9}(-3+i\sqrt{3})xz^2$. The curve R_4 is a rational nodal cubic with a node $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$. The line N and the curves R_1 and R_4 are pairwise triply tangent at $[0:1:0]$. Also the curves R_1 and R_4 intersect at each of the points $[0:0:1]$ and $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$ with intersection multiplicity 3. The line N is triply tangent to R_1 at $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$ and intersects R_4 at the same point with intersection multiplicity 3. Therefore

the configuration of curves depicted by Figure 27(d) is verified, from which the appropriate sequence of blow-ups verifies the existence of the embedding of curves with intersections given by Figure 27(b). A simple count of blow-ups shows that the resulting configuration is in $\mathbb{CP}^2\#(|G'|-1)\overline{\mathbb{CP}^2}$. The existence of the smoothing then follows from Pinkham's Theorem 2.12. □

Proof of Theorem 4.9 As before, the implication (1) $\Rightarrow$ (2) follows from general principles, (2) $\Rightarrow$ (3) is a direct consequence of Proposition 4.10 and (3) $\Rightarrow$ (1) is implied by Lemma 4.11. □

Remark 4.12 The graphs found in this case are constructed by the usual strategy of always blowing up the edge connecting the (-1)-vertex with the leaf. Notice that the existence of the smoothings has been already verified in [17, Example 8.12].

After examining all possibilities, we arrive at the

Proof of Theorem 1.6 Consider a Seifert singularity S_Γ with minimal good resolution graph having at least four legs. Once again, the existence of a rational homology disk smoothing implies the existence of a rational homology disk filling of the Milnor fillable contact structure ξ_Γ on the link Y_Γ showing the implication (1) $\Rightarrow$ (2). Suppose now that (Y_Γ, ξ_Γ) admits a rational homology disk filling. By Theorem 2.9 we get that Γ is a 4-legged graph in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$. Therefore the combination of Theorems 4.1, 4.5 and 4.9 implies both (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1), concluding the proof of the theorem. □

5 Appendix: the families $\mathcal{W}, \mathcal{N}$ and $\mathcal{M}$

For completeness, we verify the existence of rational homology disk smoothings for the singularities with resolution graphs in $\mathcal{W}, \mathcal{M}$ and $\mathcal{N}$. Notice that these results were already proved in [17] by slightly different means; here we sketch this alternative argument to unify the treatment of all cases.

The family $\mathcal{W}$

Graphs in the family $\mathcal{W}$ were defined in [17, Figure 3.] (cf. [18] for the first appearance of these plumbing trees); we depicted the graphs of this family

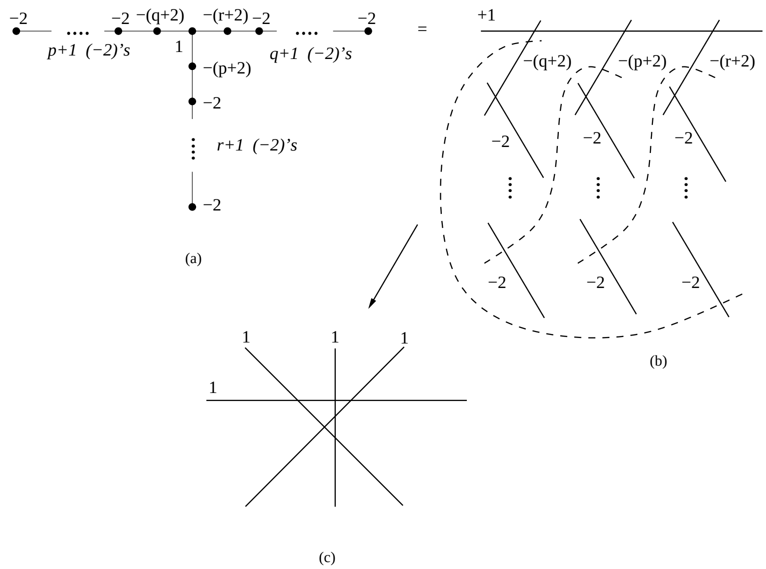

Figure 28: The dual graphs, the (-1)-curves and the configuration of curves after successively blowing down in the family $\mathcal{W}$. (The graphs in $\mathcal{W}$ are shown in Figure 1(a).)

in Figure 1(a). For the dual plumbing, see Figure 28(a). Adding three (-1)-curves to the duals as shown in (b) and successively blowing down results in the configuration shown in Figure 28(c) in the complex projective plane. Since the diagram depicts four generic lines in the complex projective plane, the existence of such a configuration is obvious. Blowing back up we get the dual configuration Γ' in $\mathbb{CP}^2\#(|\Gamma'|-1)\overline{\mathbb{CP}^2}$, which according to [16, Theorem 6.7] (cf. also Theorem 2.12) provides the existence of the rational homology disk smoothing. The same statement has been verified in [18] and in [17, Example 8.4].

The family $\mathcal{N}$

Figure 29(a) shows the dual graphs of the triply infinite family of graphs forming $\mathcal{N}$, the family given by the graphs of Figures 1(b) and (c). (Notice that the difference between the $p > 0$ and $p = 0$ case, which is apparent in the graphs of Figures 1(b) and (c), disappears when we pass to the duals.) The result of one blow-up (shown in Figure 29(b)) and then two blow-downs is shown in (c), where the parabola is tangent to the horizontal line. From this configuration (after successively blowing down the (-1)-curves, starting with the dashed

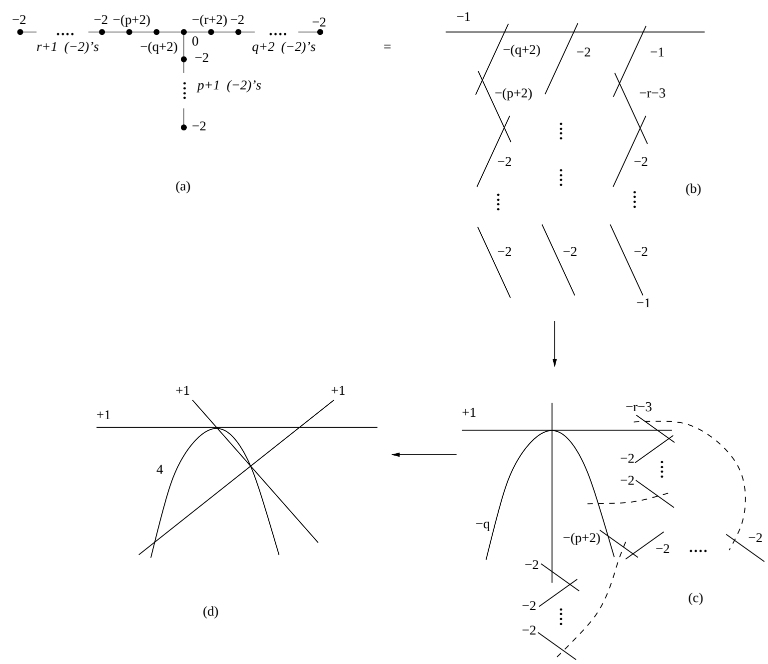

Figure 29: The duals, the (-1)-curves and the configuration of curves after successively blowing down in the family $\mathcal{N}$. (The graphs in $\mathcal{N}$ are shown by Figure 1(b) and (c).)

ones) results the configuration of a conic and three lines in $\mathbb{CP}^2$. When $p = 0$, the vertical (-1)-curve is missing in (c), and correspondingly the final configuration consists of two lines and a conic. It is elementary to give examples of a conic, a tangent line to it, and two further lines intersecting according to the diagram in Figure 29(d). The reverse of the blow-down procedure, together with Pinkham's Theorem 2.12, shows the existence of the rational homology disk smoothing. Once again, a similar argument for the existence of the rational homology disk smoothing has been presented in [17, Example 8.4].

The family $\mathcal{M}$

As usual, Figure 30(a) depicts the duals of the graphs in the triply infinite family $\mathcal{M}$, the family defined by the diagrams of Figures 1(d), (e), (f) and (g). Notice that the various degenerations $p = 0$, $r = 0$ or both (shown in Figure 1) are absorbed by the dual graphs. (For $r = 0$ the vertical (-1), together with

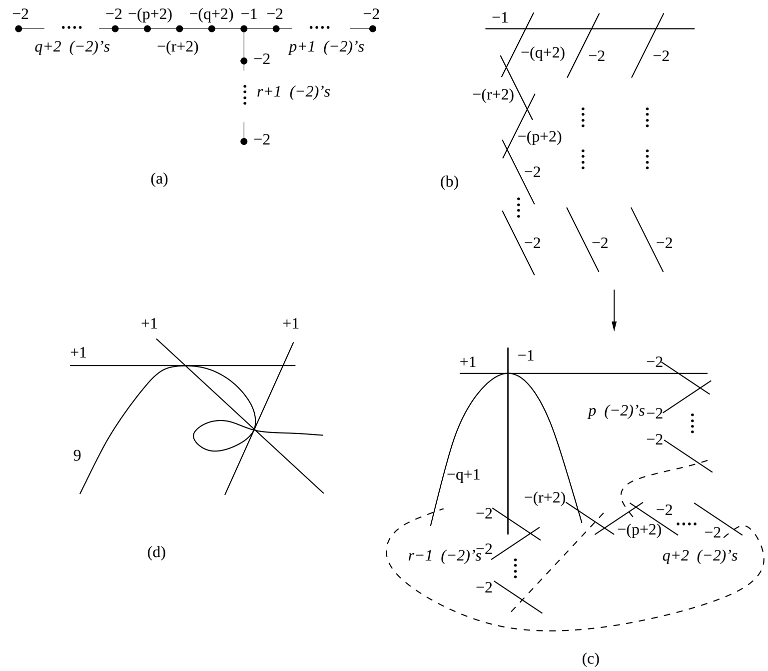

Figure 30: The duals, the (-1)-curves and the configuration of curves after successively blowing down in the family $\mathcal{M}$. (The graphs in $\mathcal{M}$ are shown by Figures 1(d), (e), (f) and (g).)

the (-2)'s hanging off it are missing, for $p=0$ the (-2)-curves attached to the horizontal $+1$ are missing, while for $p=r=0$ both these groups of curves are not there.) Successively blowing down (-1)-curves (starting with the dashed ones) results in the configuration given in (d) (again, for $p=0$ or $r=0$ a line is missing, and for $p=r=0$ two lines are not there). A cubic curve with a transverse double point — for example the one given by $\{y^2z-x^3-x^2z=0\}$ — together with a tangent at one of its inflection points (e.g., $\{z=0\}$ intersecting it at $[0:1:0]$) and the two further lines $\{x=0\}$ and $\{x+y=0\}$ provide such a configuration. Once again, this argument shows the existence of the rational homology disk filling, which was already verified in [17, Example 8.3].

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